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References for OER project MATH 140

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Chapter 2: Limits and Derivatives

Section 2.1: The Tangent and Velocity Problems

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in **Example 2.12**.)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

2.1 | A Preview of Calculus

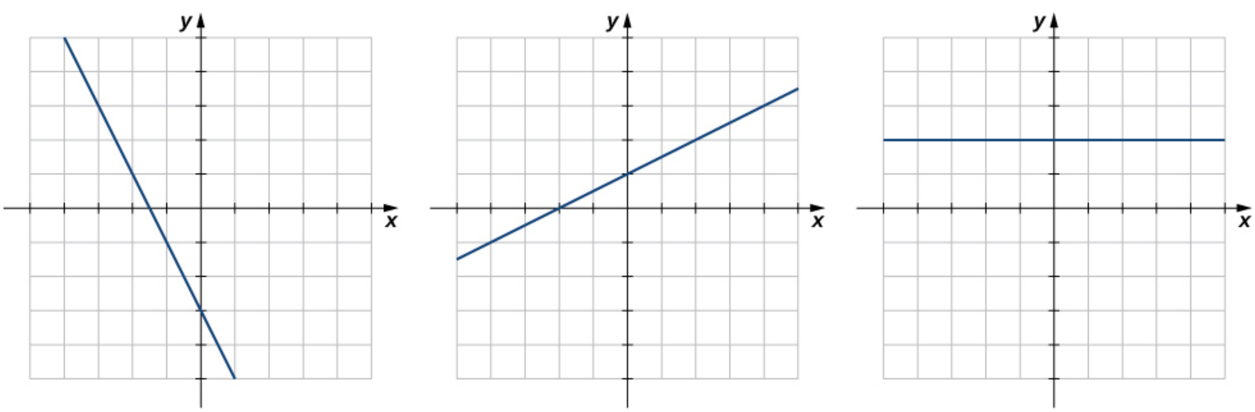
Learning Objectives

2.1.1 Describe the tangent problem and how it led to the idea of a derivative.
2.1.2 Explain how the idea of a limit is involved in solving the tangent problem.
2.1.3 Recognize a tangent to a curve at a point as the limit of secant lines.
2.1.4 Identify instantaneous velocity as the limit of average velocity over a small time interval.
2.1.5 Describe the area problem and how it was solved by the integral.
2.1.6 Explain how the idea of a limit is involved in solving the area problem.
2.1.7 Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3,\ g(x) = \frac{1}{2}x + 1,$ and $h(x) = 2,$ shown in **Figure 2.2**.



$f(x) = -2x - 3$ $\quad$ $g(x) = \frac{x}{2} + 1$ $\quad$ $h(x) = 2$

Figure 2.2 The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3,$ we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This rate of change is determined by the slope (−2) of the line. Similarly, the slope of 1/2 in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of 1/2 unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (**Figure 2.3**). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?

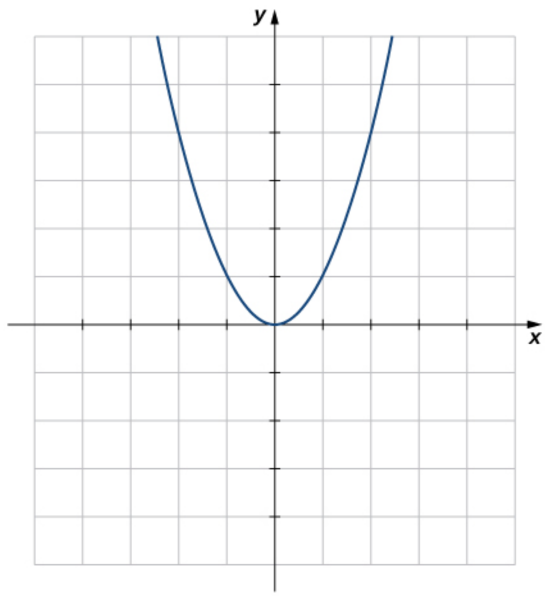


Figure 2.3 The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. **Figure 2.4** shows a secant line to a function $f(x)$ at a point $(a, f(a))$.

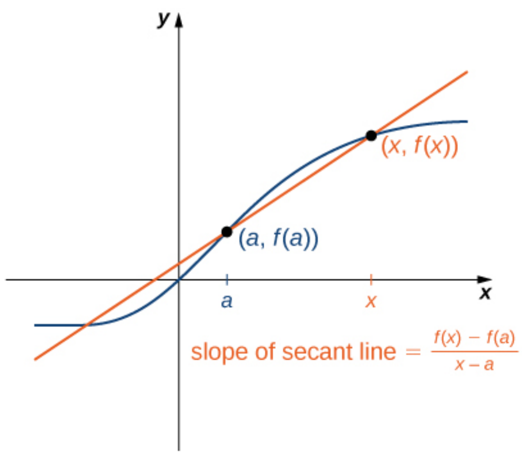


Figure 2.4 The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}. \quad (2.1)$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in **Figure 2.5**, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.

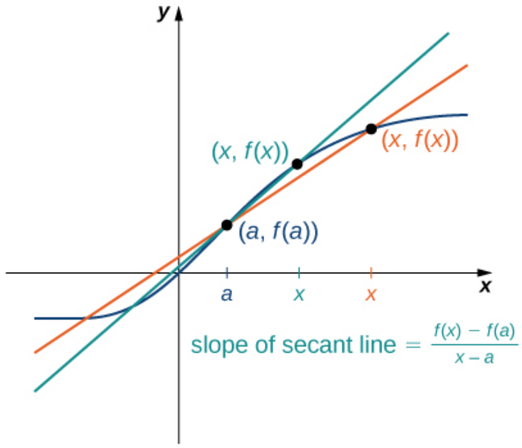


Figure 2.5 As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (**Figure 2.6**). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): **Math Insight (http://www.openstaxcollege.org/l/20_mathinsight)** .

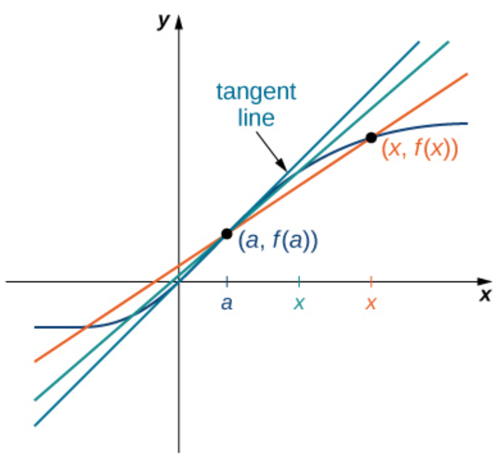


Figure 2.6 Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.1

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$

b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution

Use the formula for the slope of a secant line from the definition.

a. $m_{\text{sec}} = \frac{4 - 1}{2 - 1} = 3$

b. $m_{\text{sec}} = \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (**Figure 2.7**).

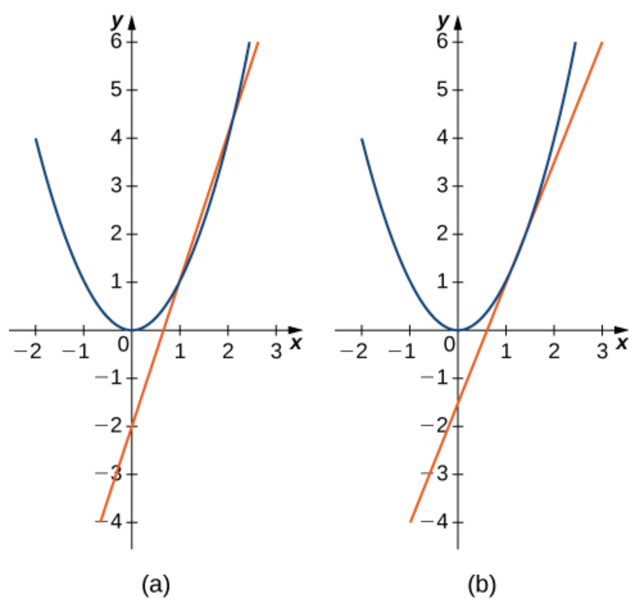


Figure 2.7 The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

2.1 Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $\left(\frac{5}{4}, \frac{25}{16}\right)$ on the graph of $f(x) = x^2$.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time *t*. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Definition

Let $s(t)$ be the position of an object moving along a coordinate axis at time *t*. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \quad (2.2)$$

As *t* is chosen closer to *a*, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point *a*, we let the *x*-values approach *a* in the slope of the secant line. Similarly, to find the instantaneous velocity at time *a*, we let the *t*-values approach *a* in the average velocity. This process of letting *x* or *t* approach *a* in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Definition

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of *t* become closer to *a*, provided such a value exists.

Example 2.2 illustrates this concept of limits and average velocity.

Example 2.2

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground *t* seconds later (for $0 \le t \le 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$

b. $[0.5, 0.51]$

Solution

Substitute the data into the formula for the definition of average velocity.

a. $v_{\text{ave}} = \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$

b. $v_{\text{ave}} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

2.2 An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

FALLING TOMATO

Suppose we drop a tomato from the top of a 100 foot building (Fig. 5) and time its fall.

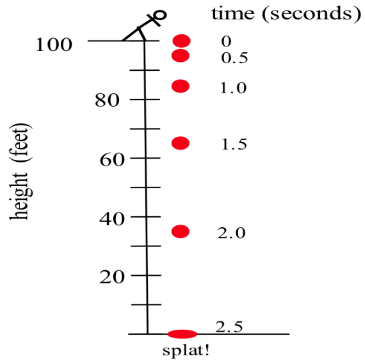


Fig. 5

time (s)	height (ft)
0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0

Some questions are easy to answer directly from the table:

(a) How long did it take for the tomato to drop 100 feet? (2.5 seconds)

(b) How far did the tomato fall during the first second? (100 – 84 = 16 feet)

(c) How far did the tomato fall during the last second? (64 – 0 = 64 feet)

(d) How far did the tomato fall between t = .5 and t = 1? (96 – 84 = 12 feet)

Some other questions require a little calculation:

(e) What was the average velocity of the tomato during its fall?

$$\textbf{Average velocity} = \frac{\textbf{distance fallen}}{\textbf{total time}} = \frac{\boldsymbol{\Delta}\textbf{position}}{\boldsymbol{\Delta}\textbf{time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40 \text{ ft/s} .$$

(f) What was the average velocity between t=1 and t=2 seconds?

$$\text{Average velocity} = \frac{\Delta \text{position}}{\Delta \text{time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48 \text{ ft/s} .$$

Some questions are more difficult.

(g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, –40 ft/s . But the tomato fell slowly at the beginning and rapidly near the end so the "–40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at t=1 by calculating the average velocities over a short time interval near t = 1 . The average velocity between t = 0.5 and t = 1 is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24$ ft/s, and the average velocity between t = 1 and t = 1.5 is $\frac{-20 \text{ feet}}{.5 \text{ s}} = -40$ ft/s so we can be reasonably sure that the instantaneous velocity is between –24 ft/s and –40 ft/s.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is $\frac{\Delta \text{position}}{\Delta \text{time}}$, which is the slope of the **secant line** through two points on the graph of height versus time (Fig. 6). The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.

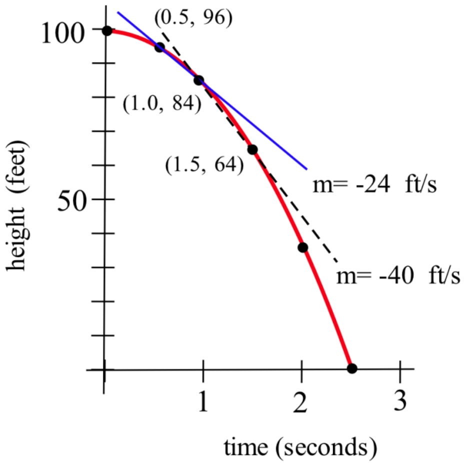


Fig. 6

Average velocity $= \frac{\Delta \text{position}}{\Delta \text{time}}$

= slope of the secant line through 2 points.

Instantaneous velocity = slope of the line tangent to the graph.

Practice 3: Estimate the velocity of the tomato 2 seconds after it was dropped.

Source: Hoffman; Contemporary Calculus; 2016 , section 1.0, pg 8

Practice 3: The average velocity between t = 1.5 and t = 2.0 is $\dfrac{36 - 64 \text{ feet}}{2.0 - 1.5 \text{ sec}} = -56$ feet per second.

The average velocity between t = 2.0 and t = 2.5 is $\dfrac{0 - 36 \text{ feet}}{2.5 - 2.0 \text{ sec}} = -72$ feet per second.

The velocity **at** t = 2.0 is somewhere between –56 ft/sec and –72 ft/sec, probably about the middle of this interval: $\dfrac{(-56) + (-72)}{2} = \mathbf{-64 \text{ ft/sec}}$.

Section 2.2: The Limit of a Function

2.2 | The Limit of a Function

Learning Objectives

- **2.2.1** Using correct notation, describe the limit of a function.
- **2.2.2** Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- **2.2.3** Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- **2.2.4** Define one-sided limits and provide examples.
- **2.2.5** Explain the relationship between one-sided and two-sided limits.
- **2.2.6** Using correct notation, describe an infinite limit.
- **2.2.7** Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

$$f(x) = \frac{x^2 - 4}{x - 2},\ g(x) = \frac{|x - 2|}{x - 2},\ \text{and } h(x) = \frac{1}{(x - 2)^2},$$

which are shown in **Figure 2.12**. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

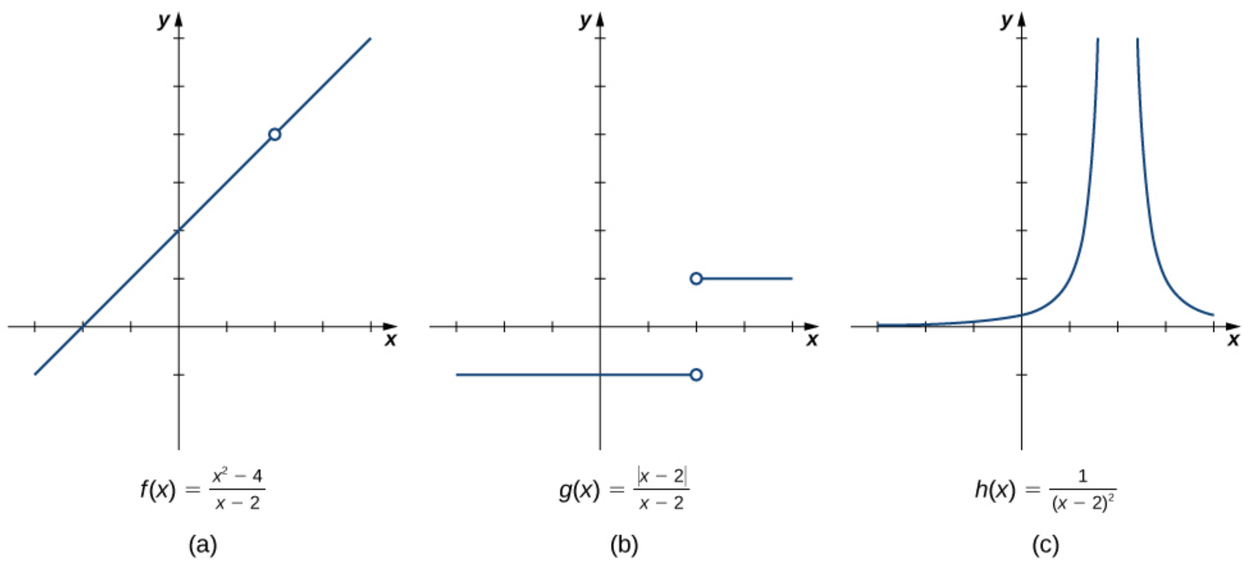


Figure 2.12 These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in **Figure 2.12**. As the values of *x* approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as *x* approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{2.3}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table 2.1 demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Table 2.1 Table of Functional Values for $\lim_{x \to a} f(x)$

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1), f(a - 0.01), f(a - 0.001)., f(a - 0.0001),$ and so on, and $f(a + 0.1), f(a + 0.01), f(a + 0.001), f(a + 0.0001),$ and so on. (*Note*: Although we have chosen the x-values $a \pm 0.1, a \pm 0.01, a \pm 0.001, a \pm 0.0001,$ and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L.$ We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in **Example 2.4**.

Example 2.4

Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x) = (\sin x)/x$ for the values of x listed in **Table 2.2**.

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
−0.1	0.998334166468		0.1	0.998334166468
−0.01	0.999983333417		0.01	0.999983333417
−0.001	0.999999833333		0.001	0.999999833333
−0.0001	0.999999998333		0.0001	0.999999998333

Table 2.2
Table of Functional Values for $\lim_{x \to 0} \frac{\sin x}{x}$

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{(\sin x)}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. A calculator-or computer-generated graph of $f(x) = \frac{(\sin x)}{x}$ would be similar to that shown in **Figure 2.13**, and it confirms our estimate.

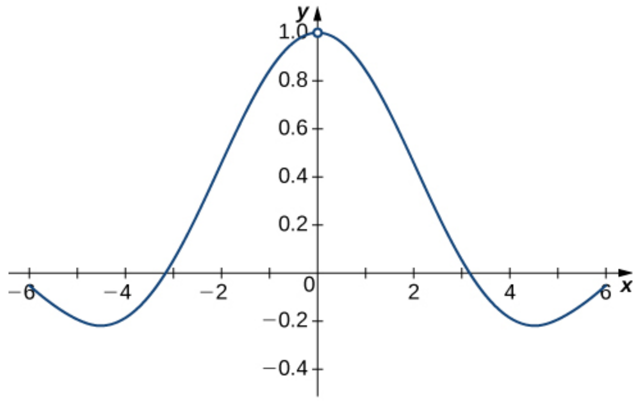


Figure 2.13 The graph of $f(x) = (\sin x)/x$ confirms the estimate from **Table 2.2**.

Example 2.5

Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ using a table of functional values.

Solution

As before, we use a table—in this case, **Table 2.3**—to list the values of the function for the given values of x.

x	$\frac{\sqrt{x}-2}{x-4}$		x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Table 2.3
Table of Functional Values for $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that

$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = 0.25$. We confirm this estimate using the graph of $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ shown in **Figure 2.14**.

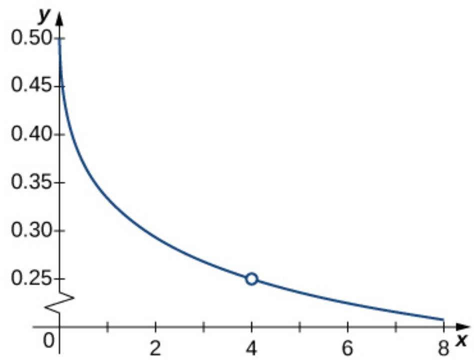


Figure 2.14 The graph of $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ confirms the estimate from **Table 2.3**.

2.4 Estimate $\lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1}$ using a table of functional values. Use a graph to confirm your estimate.

At this point, we see from **Example 2.4** and **Example 2.5** that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In **Example 2.6**, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example 2.6

Evaluating a Limit Using a Graph

For $g(x)$ shown in **Figure 2.15**, evaluate $\lim_{x \to -1} g(x)$.

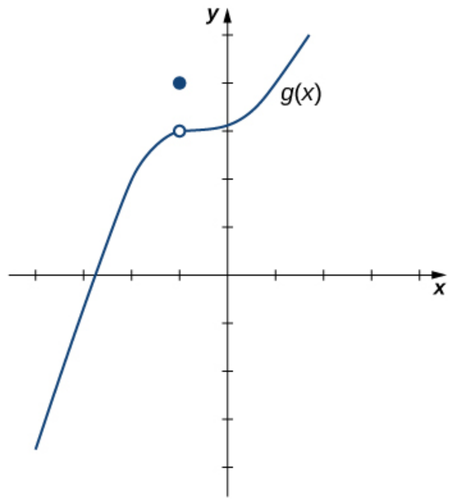


Figure 2.15 The graph of $g(x)$ includes one value not on a smooth curve.

Solution

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on **Example 2.6**, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

2.5 Use the graph of $h(x)$ in **Figure 2.16** to evaluate $\lim_{x \to 2} h(x)$, if possible.

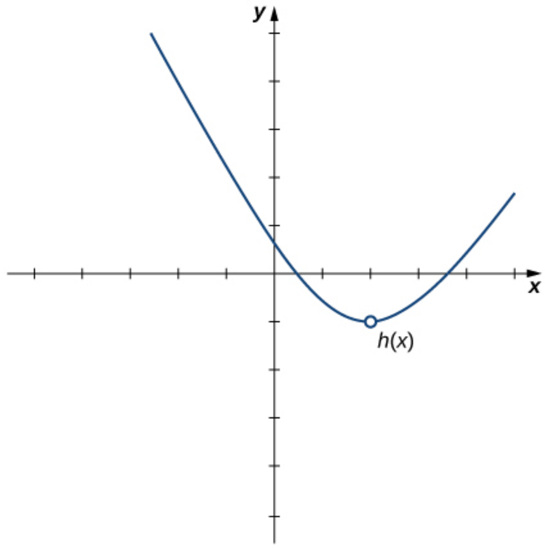


Figure 2.16

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Theorem 2.1: Two Important Limits

Let a be a real number and c be a constant.

i. $$\lim_{x \to a} x = a \quad (2.4)$$

ii. $$\lim_{x \to a} c = c \quad (2.5)$$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.

ii. For the second limit, consider **Table 2.4**.

x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table 2.4 Table of Functional Values for $\lim_{x \to a} c = c$

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim_{x \to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.7

Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x \to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.5 lists values for the function $\sin(1/x)$ for the given values of x.

x	$\sin\left(\frac{1}{x}\right)$		x	$\sin\left(\frac{1}{x}\right)$
−0.1	0.544021110889		0.1	−0.544021110889
−0.01	0.50636564111		0.01	−0.50636564111
−0.001	−0.8268795405312		0.001	0.826879540532
−0.0001	0.305614388888		0.0001	−0.305614388888
−0.00001	−0.035748797987		0.00001	0.035748797987
−0.000001	0.349993504187		0.000001	−0.349993504187

Table 2.5
Table of Functional Values for $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim_{x \to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate “does not exist” as DNE. Thus, we would write $\lim_{x \to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in **Figure 2.17** and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between −1 and 1 as x approaches 0.

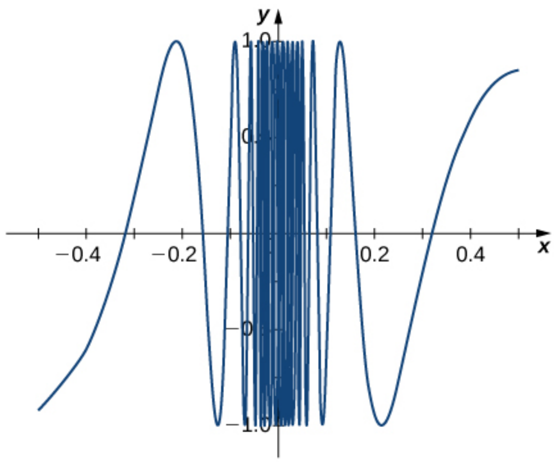


Figure 2.17 The graph of $f(x) = \sin(1/x)$ oscillates rapidly between −1 and 1 as x approaches 0.

2.6 Use a table of functional values to evaluate $\lim_{x \to 2} \frac{|x^2 - 4|}{x - 2}$, if possible.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see **Figure 2.12**(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or *the negative side of* 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches −1. Mathematically, we say that the limit as x approaches 2 from the left is −1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or *from the positive side*), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition

We define two types of **one-sided limits**.

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form z, and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L. \quad (2.6)$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L. \quad (2.7)$$

Example 2.8

Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again **Table 2.6**. Observe that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

x	$f(x) = x+1$		x	$f(x) = x^2-4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table 2.6
Table of Functional Values for $f(x) = \begin{cases} x+1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. **Figure 2.18** shows a graph of $f(x)$ and reinforces our conclusion about these limits.

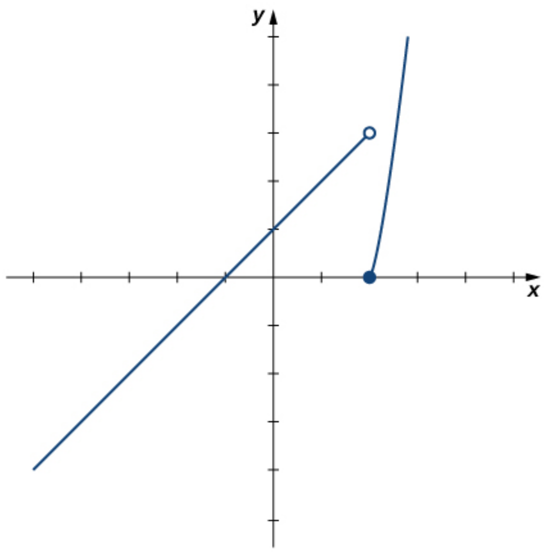


Figure 2.18 The graph of $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

2.7 Use a table of functional values to estimate the following limits, if possible.

a. $\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$

b. $\lim_{x \to 2^+} \frac{|x^2 - 4|}{x - 2}$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in **Relating One-Sided and Two-Sided Limits**.

Theorem 2.2: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. Then,

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see **Figure 2.12**(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define **infinite limits** as follows:

Definition

We define three types of **infinite limits**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty. \tag{2.8}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty. \tag{2.9}$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty. \tag{2.10}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty. \tag{2.11}$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a.

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty. \tag{2.12}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty. \tag{2.13}$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.9

Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \frac{1}{x}$

b. $\lim_{x \to 0^+} \frac{1}{x}$

c. $\lim_{x \to 0} \frac{1}{x}$

Solution

Begin by constructing a table of functional values.

x	$\frac{1}{x}$		x	$\frac{1}{x}$
−0.1	−10		0.1	10
−0.01	−100		0.01	100
−0.001	−1000		0.001	1000
−0.0001	−10,000		0.0001	10,000
−0.00001	−100,000		0.00001	100,000
−0.000001	−1,000,000		0.000001	1,000,000

Table 2.7
Table of Functional Values for $f(x) = \frac{1}{x}$

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x \to 0} \frac{1}{x} \text{ DNE.}$$

The graph of $f(x) = 1/x$ in **Figure 2.19** confirms these conclusions.

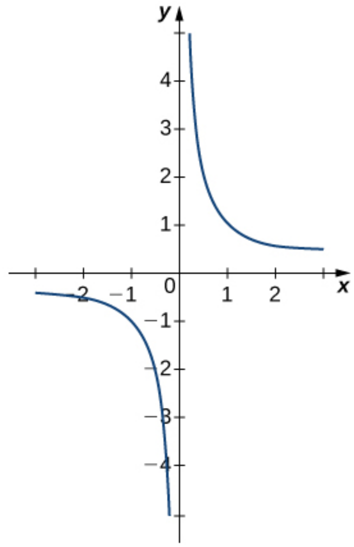


Figure 2.19 The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

2.8 Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \frac{1}{x^2}$

b. $\lim\limits_{x \to 0^+} \frac{1}{x^2}$

c. $\lim\limits_{x \to 0} \frac{1}{x^2}$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (**Figure 2.20**). These limits are summarized in **Infinite Limits from Positive Integers**.

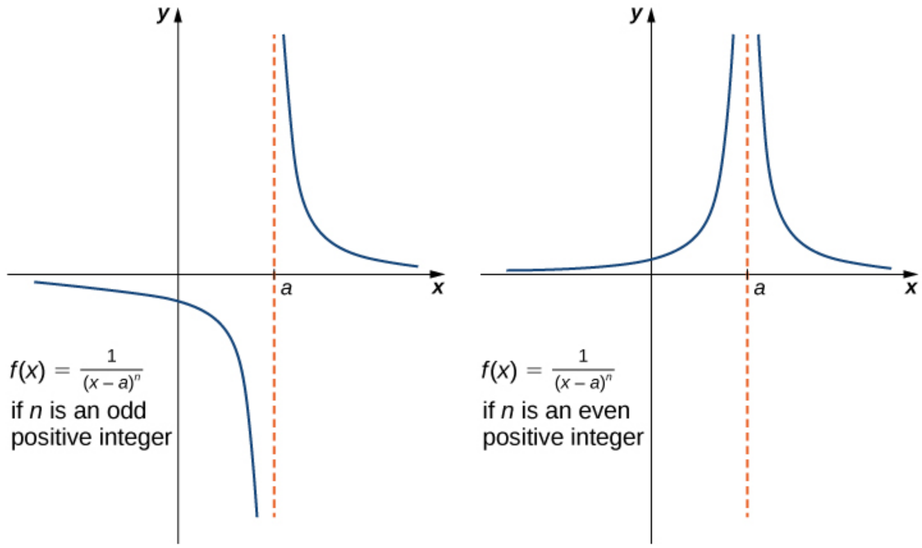


Figure 2.20 The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

Theorem 2.3: Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

Definition

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

$$\begin{array}{rcl} \lim_{x \to a^-} f(x) & = & +\infty \text{ or } -\infty \\ \lim_{x \to a^+} f(x) & = & +\infty \text{ or } -\infty \\ & \text{or} & \\ \lim_{x \to a} f(x) & = & +\infty \text{ or } -\infty \end{array}$$

Example 2.10

Finding a Vertical Asymptote

Evaluate each of the following limits using **Infinite Limits from Positive Integers**. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim\limits_{x \to -3^-} \frac{1}{(x+3)^4}$

b. $\lim\limits_{x \to -3^+} \frac{1}{(x+3)^4}$

c. $\lim\limits_{x \to -3} \frac{1}{(x+3)^4}$

Solution

We can use **Infinite Limits from Positive Integers** directly.

a. $\lim\limits_{x \to -3^-} \frac{1}{(x+3)^4} = +\infty$

b. $\lim\limits_{x \to -3^+} \frac{1}{(x+3)^4} = +\infty$

c. $\lim\limits_{x \to -3} \frac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

2.9 Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim\limits_{x \to 2^-} \frac{1}{(x-2)^3}$

b. $\lim\limits_{x \to 2^+} \frac{1}{(x-2)^3}$

c. $\lim\limits_{x \to 2} \frac{1}{(x-2)^3}$

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.11

Behavior of a Function at Different Points

Use the graph of $f(x)$ in **Figure 2.21** to determine each of the following values:

a. $\lim\limits_{x \to -4^-} f(x)$; $\lim\limits_{x \to -4^+} f(x)$; $\lim\limits_{x \to -4} f(x)$; $f(-4)$

b. $\lim_{x \to -2^-} f(x)$; $\lim_{x \to -2^+} f(x)$; $\lim_{x \to -2} f(x)$; $f(-2)$

c. $\lim_{x \to 1^-} f(x)$; $\lim_{x \to 1^+} f(x)$; $\lim_{x \to 1} f(x)$; $f(1)$

d. $\lim_{x \to 3^-} f(x)$; $\lim_{x \to 3^+} f(x)$; $\lim_{x \to 3} f(x)$; $f(3)$

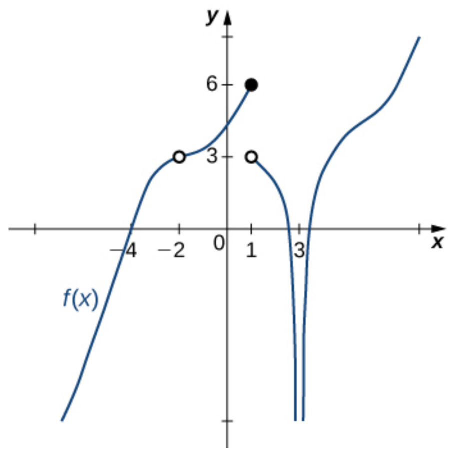


Figure 2.21 The graph shows $f(x)$.

Solution

Using **Infinite Limits from Positive Integers** and the graph for reference, we arrive at the following values:

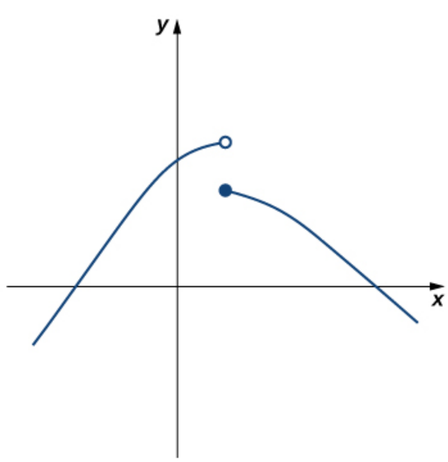
a. $\lim_{x \to -4^-} f(x) = 0$; $\lim_{x \to -4^+} f(x) = 0$; $\lim_{x \to -4} f(x) = 0$; $f(-4) = 0$

b. $\lim_{x \to -2^-} f(x) = 3.$; $\lim_{x \to -2^+} f(x) = 3$; $\lim_{x \to -2} f(x) = 3$; $f(-2)$ is undefined

c. $\lim_{x \to 1^-} f(x) = 6$; $\lim_{x \to 1^+} f(x) = 3$; $\lim_{x \to 1} f(x)$ DNE; $f(1) = 6$

d. $\lim_{x \to 3^-} f(x) = -\infty$; $\lim_{x \to 3^+} f(x) = -\infty$; $\lim_{x \to 3} f(x) = -\infty$; $f(3)$ is undefined

2.10 Evaluate $\lim_{x \to 1} f(x)$ for $f(x)$ shown here:



Section 2.3: Calculating Limits Using the Limit Laws

2.3 | The Limit Laws

Learning Objectives

2.3.1 Recognize the basic limit laws.
2.3.2 Use the limit laws to evaluate the limit of a function.
2.3.3 Evaluate the limit of a function by factoring.
2.3.4 Use the limit laws to evaluate the limit of a polynomial or rational function.
2.3.5 Evaluate the limit of a function by factoring or by using conjugates.
2.3.6 Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in **Two Important Limits** and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.4: Basic Limit Results

For any real number a and any constant c,

i. $$\lim_{x \to a} x = a \tag{2.14}$$

ii. $$\lim_{x \to a} c = c \tag{2.15}$$

Example 2.13

Evaluating a Basic Limit

Evaluate each of the following limits using **Basic Limit Results**.

a. $\lim_{x \to 2} x$

b. $\lim_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim_{x \to 2} x = 2$.

b. The limit of a constant is that constant: $\lim_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.5: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example 2.14

Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim_{x \to -3}(4x + 2)$.

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3}(4x + 2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example 2.15

Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned}\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4} &= \frac{\lim_{x \to 2}(2x^2 - 3x + 1)}{\lim_{x \to 2}(x^3 + 4)} && \text{Apply the quotient law, making sure that. } (2)^3 + 4 \neq 0 \\ &= \frac{2 \cdot \lim_{x \to 2} x^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\lim_{x \to 2} x^3 + \lim_{x \to 2} 4} && \text{Apply the sum law and constant multiple law.} \\ &= \frac{2 \cdot \left(\lim_{x \to 2} x\right)^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\left(\lim_{x \to 2} x\right)^3 + \lim_{x \to 2} 4} && \text{Apply the power law.} \\ &= \frac{2(4) - 3(2) + 1}{(2)^3 + 4} = \frac{1}{4}. && \text{Apply the basic limit laws and simplify.}\end{aligned}$$

2.11 Use the limit laws to evaluate $\lim_{x \to 6} (2x - 1)\sqrt{x + 4}$. In each step, indicate the limit law applied.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of *a* and for all rational functions at all values of *a* for which the rational function is defined.

Theorem 2.6: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let *a* be a real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}\lim_{x \to a} p(x) &= \lim_{x \to a}\left(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\right) \\ &= c_n \left(\lim_{x \to a} x\right)^n + c_{n-1}\left(\lim_{x \to a} x\right)^{n-1} + \cdots + c_1\left(\lim_{x \to a} x\right) + \lim_{x \to a} c_0 \\ &= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \\ &= p(a).\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Example 2.16 applies this result.

Example 2.16

Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

2.12 Evaluate $\lim_{x \to -2} \left(3x^3 - 2x + 7\right)$.

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \to a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

To understand this idea better, consider the limit $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

The function

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in **Figure 2.24**.

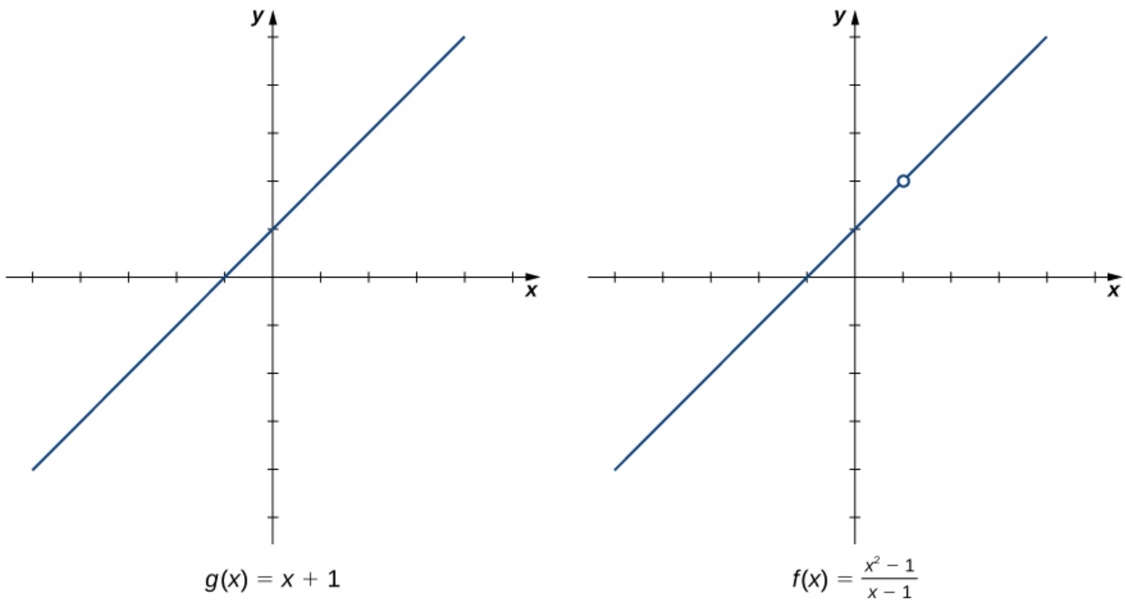


Figure 2.24 The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\begin{aligned}\lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x + 1) \\ &= 2.\end{aligned}$$

The limit has the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form 0/0

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. **Example 2.17** illustrates the factor-and-cancel technique; **Example 2.18** shows multiplying by a conjugate. In **Example 2.19**, we look at simplifying a complex fraction.

Example 2.17

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution

Step 1. The function $f(x) = \frac{x^2 - 3x}{2x^2 - 5x - 3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(2x + 1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2 - 3x}{2x^2 - 5x - 3} = \frac{x}{2x + 1}$. Therefore,

$$\lim_{x \to 3} \frac{x(x - 3)}{(x - 3)(2x + 1)} = \lim_{x \to 3} \frac{x}{2x + 1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x \to 3} \frac{x}{2x + 1} = \frac{3}{7}.$$

2.13 Evaluate $\lim_{x \to -3} \frac{x^2 + 4x + 3}{x^2 - 9}$.

Example 2.18

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x \to -1} \frac{\sqrt{x + 2} - 1}{x + 1}$.

Solution

Step 1. $\frac{\sqrt{x + 2} - 1}{x + 1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x + 2} + 1$, the conjugate of $\sqrt{x + 2} - 1$, on the numerator and denominator:

$$\lim_{x \to -1} \frac{\sqrt{x + 2} - 1}{x + 1} = \lim_{x \to -1} \frac{\sqrt{x + 2} - 1}{x + 1} \cdot \frac{\sqrt{x + 2} + 1}{\sqrt{x + 2} + 1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x + 1)$ in the denominator cancels out in the end:

$$= \lim_{x \to -1} \frac{x + 1}{(x + 1)\left(\sqrt{x + 2} + 1\right)}.$$

Step 3. Then we cancel:

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$

2.14 Evaluate $\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$.

Example 2.19

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

Solution

Step 1. $\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$= \lim_{x \to 1} \frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$

2.15 Evaluate $\lim\limits_{x \to -3} \dfrac{\frac{1}{x+2} + 1}{x+3}$.

Example 2.20 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.20

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim\limits_{x \to 0} \left(\dfrac{1}{x} + \dfrac{5}{x(x-5)} \right)$.

Solution

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x} + \frac{5}{x(x-5)} = \frac{x-5+5}{x(x-5)} = \frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right) = \lim_{x \to 0} \frac{x}{x(x-5)} = \lim_{x \to 0} \frac{1}{x-5} = -\frac{1}{5}.$$

2.16 Evaluate $\lim\limits_{x \to 3} \left(\dfrac{1}{x-3} - \dfrac{4}{x^2 - 2x - 3} \right)$.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim\limits_{x \to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a); for a limit of the form $\lim\limits_{x \to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c). **Example 2.21** illustrates this point.

Example 2.21

Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim\limits_{x \to 3^-} \sqrt{x-3}$

b. $\lim_{x \to 3^+} \sqrt{x-3}$

Solution

Figure 2.25 illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.

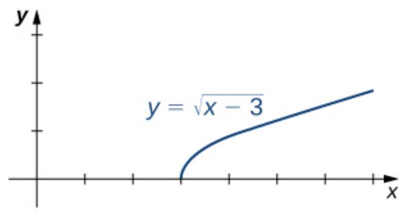


Figure 2.25 The graph shows the function $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x \to 3^-} \sqrt{x-3}$ does not exist.

b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x \to 3^+} \sqrt{x-3} = 0$.

In Example 2.22 we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example 2.22

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x-3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

Solution

Figure 2.26 illustrates the function $f(x)$ and aids in our understanding of these limits.

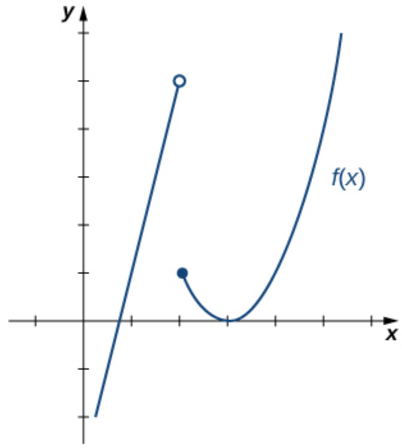


Figure 2.26 This graph shows a function $f(x)$.

a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x - 3) = 5.$$

b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} (x - 3)^2 = 1.$$

c. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$, we conclude that $\lim_{x \to 2} f(x)$ does not exist.

2.17 Graph $f(x) = \begin{cases} -x - 2 \text{ if } x < -1 \\ 2 \text{ if } x = -1 \\ x^3 \text{ if } x > -1 \end{cases}$ and evaluate $\lim_{x \to -1^-} f(x)$.

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.23

Evaluating a Limit of the Form $K/0, K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \frac{x - 3}{x^2 - 2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches −1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}.$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

Step 3. $\lim_{x \to 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

2.18 Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$.

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point *a* that is unknown, between two functions having a common known limit at *a*. Figure 2.27 illustrates this idea.

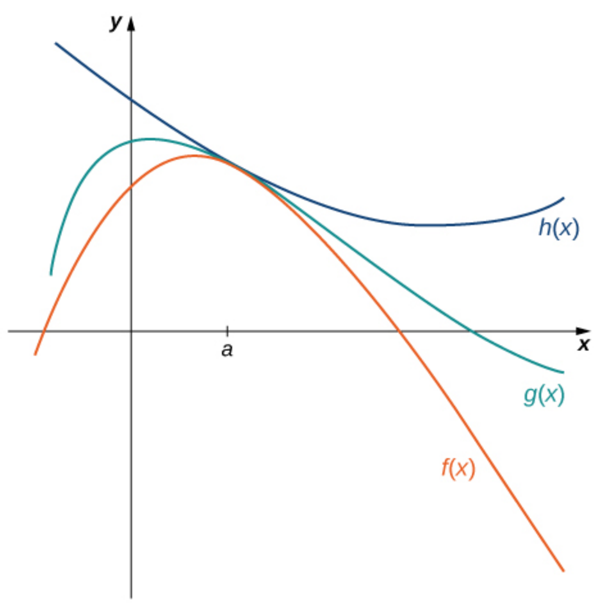


Figure 2.27 The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Theorem 2.7: The Squeeze Theorem

Let $f(x), g(x),$ and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 2.24

Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution

Because $-1 \leq \cos x \leq 1$ for all x, we have $-|x| \leq x \cos x \leq |x|$. Since $\lim_{x \to 0} (-|x|) = 0 = \lim_{x \to 0} |x|$, from the squeeze theorem, we obtain $\lim_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -|x|$, $g(x) = x \cos x$, and $h(x) = |x|$ are shown in Figure 2.28.

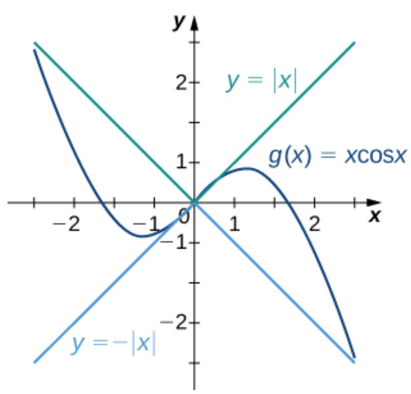


Figure 2.28 The graphs of $f(x), g(x),$ and $h(x)$ are shown around the point $x = 0$.

2.19 Use the squeeze theorem to evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \to 0} \sin \theta$. Consider the unit circle shown in Figure 2.29. In the figure, we see that $\sin \theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, $0 < \sin \theta < \theta$.

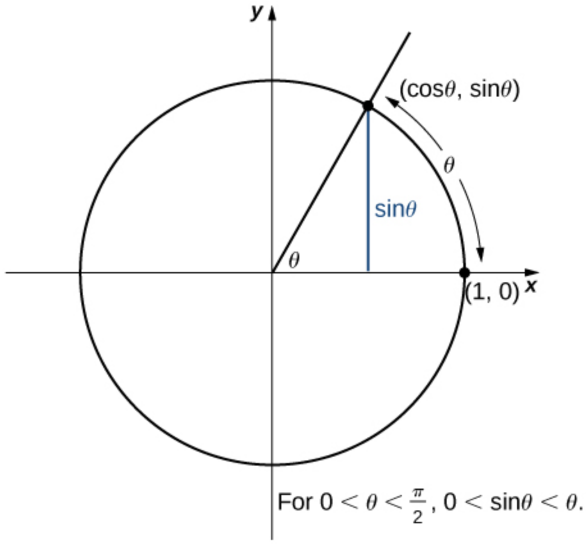


Figure 2.29 The sine function is shown as a line on the unit circle.

Because $\lim_{\theta \to 0^+} 0 = 0$ and $\lim_{\theta \to 0^+} \theta = 0$, by using the squeeze theorem we conclude that

$$\lim_{\theta \to 0^+} \sin\theta = 0.$$

To see that $\lim_{\theta \to 0^-} \sin\theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0$, $0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin\theta < -\theta$. It follows that $0 > \sin\theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim_{\theta \to 0^+} \sin\theta = 0$ and $\lim_{\theta \to 0^-} \sin\theta = 0$,

$$\lim_{\theta \to 0} \sin\theta = 0. \quad (2.16)$$

Next, using the identity $\cos\theta = \sqrt{1 - \sin^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

$$\lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2\theta} = 1. \quad (2.17)$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$. To evaluate this limit, we use the unit circle in **Figure 2.30**. Notice that this figure adds one additional triangle to **Figure 2.30**. We see that the length of the side opposite angle θ in this new triangle is $\tan\theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, $\sin\theta < \theta < \tan\theta$.

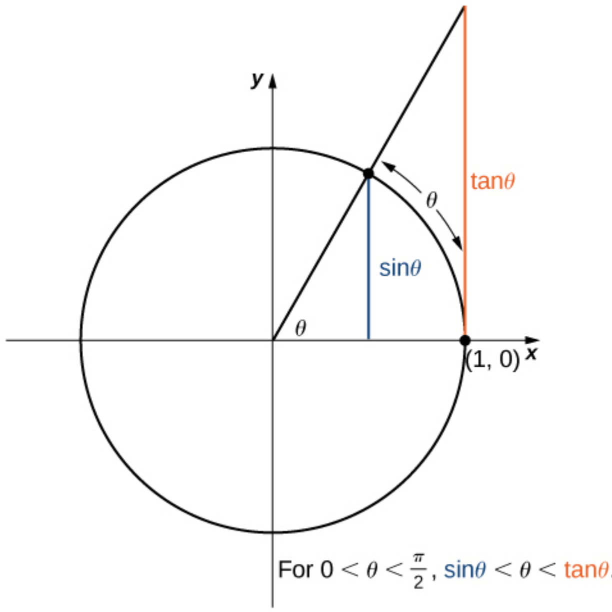


Figure 2.30 The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim\limits_{\theta \to 0^+} 1 = 1 = \lim\limits_{\theta \to 0^+} \cos\theta,$ we conclude that $\lim\limits_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$ By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta \to 0^-} \sin\theta = 0,$ we can show that $\lim\limits_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1.$ Thus,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1. \qquad (2.18)$$

In **Example 2.25** we use this limit to establish $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$ This limit also proves useful in later chapters.

Example 2.25

Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta}.$

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \cdot \frac{1 + \cos\theta}{1 + \cos\theta} \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1 + \cos\theta} \\
&= 1 \cdot \frac{0}{2} = 0.
\end{aligned}$$

Therefore,

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0. \tag{2.19}$$

2.20 Evaluate $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta}$.

Section 2.4: The Precise Definition of a Limit

This video provides an intuitive explanation of the Epsilon-Delta definition for limits: Khan Academy - Epsilon-Delta Limit Definition Part 1

This video provides an explanation of how to calculate an infinite limit at a point. This was covered by the examples in Stewart, but not in the OpenStax textbook. Dr Peyam - Epsilon delta limit (Example 3): Infinite limit at a point

2.5 | The Precise Definition of a Limit

Learning Objectives

2.5.1 Describe the epsilon-delta definition of a limit.
2.5.2 Apply the epsilon-delta definition to find the limit of a function.
2.5.3 Describe the epsilon-delta definitions of one-sided limits and infinite limits.
2.5.4 Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a-b|$.

- The statement $|f(x)-L|<\varepsilon$ may be interpreted as: *The distance between* $f(x)$ *and L is less than ε.*
- The statement $0<|x-a|<\delta$ may be interpreted as: $x \neq a$ *and the distance between x and a is less than δ.*

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x)-L|<\varepsilon$ is equivalent to the statement $L-\varepsilon<f(x)<L+\varepsilon$.
- The statement $0<|x-a|<\delta$ is equivalent to the statement $a-\delta<x<a+\delta$ and $x \neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x)=L$$

if, for every $\varepsilon>0$, there exists a $\delta>0$, such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a *universal quantifier* (for every $\varepsilon>0$), an *existential quantifier* (there exists a $\delta>0$), and, last, a *conditional statement* (if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$). Let’s take a look at **Table 2.9**, which breaks down the definition and translates each part.

Definition	Translation
1. For every $\varepsilon > 0,$	1. For every positive distance *ε* from *L*,
2. there exists a $\delta > 0,$	2. There is a positive distance δ from *a*,
3. such that	3. such that
4. if $0 < \lvert x - a \rvert < \delta,$ then $\lvert f(x) - L \rvert < \varepsilon.$	4. if *x* is closer than δ to *a* and $x \neq a,$ then $f(x)$ is closer than *ε* to *L*.

Table 2.9 Translation of the Epsilon-Delta Definition of the Limit

We can get a better handle on this definition by looking at the definition geometrically. **Figure 2.39** shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x),$ a number a, and a limit L at a. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.

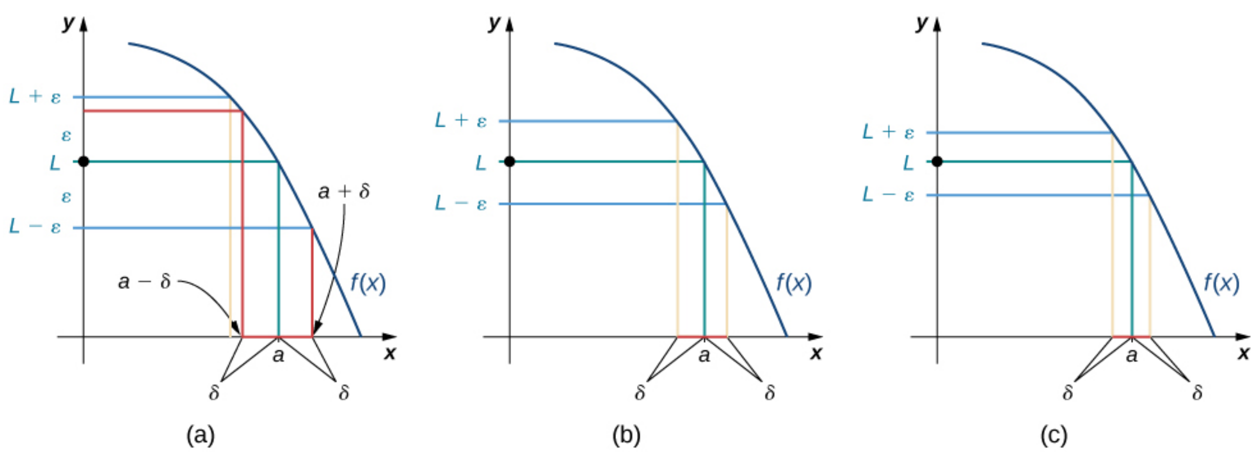


Figure 2.39 These graphs show possible values of δ, given successively smaller choices of ε.

Visit the following applet to experiment with finding values of δ for selected values of ε:

- **The epsilon-delta definition of limit (http://www.openstaxcollege.org/l/20_epsilondelt)**

Example 2.39 shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example 2.39

Proving a Statement about the Limit of a Specific Function

Prove that $\lim\limits_{x \to 1} (2x + 1) = 3.$

Solution

Let $\varepsilon > 0$.

The first part of the definition begins "For every $\varepsilon > 0$." This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$." The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x+1) - 3| < \varepsilon$, we begin by manipulating this expression: $|(2x+1) - 3| < \varepsilon$ is equivalent to $|2x - 2| < \varepsilon$, which in turn is equivalent to $|2||x - 1| < \varepsilon$. Last, this is equivalent to $|x - 1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. **Figure 2.40** demonstrates how this is done.

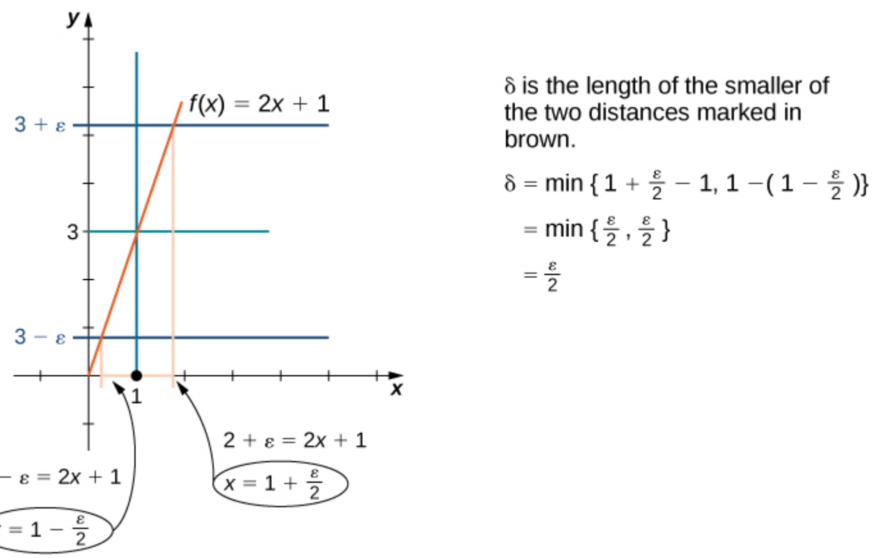


Figure 2.40 This graph shows how we find δ geometrically.

Assume $0 < |x - 1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x - 1| < \delta$, then $|(2x+1) - 3| < \varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$$\begin{aligned} |(2x+1)-3| &= |2x-2| && \text{property of absolute value} \\ &= |2(x-1)| \\ &= |2||x-1| && |2| = 2 \\ &= 2|x-1| \\ &< 2\cdot\delta && \text{here's where we use the assumption that } 0 < |x-1| < \delta \\ &= 2\cdot\frac{\varepsilon}{2} = \varepsilon && \text{here's where we use our choice of } \delta = \varepsilon/2 \end{aligned}$$

Analysis

In this part of the proof, we started with $|(2x+1)-3|$ and used our assumption $0 < |x-1| < \delta$ in a key part of the chain of inequalities to get $|(2x+1)-3|$ to be less than ε. We could just as easily have manipulated the assumed inequality $0 < |x-1| < \delta$ to arrive at $|(2x+1)-3| < \varepsilon$ as follows:

$$\begin{aligned}
0 < |x-1| < \delta \; &\Rightarrow |x-1| < \delta \\
&\Rightarrow -\delta < x-1 < \delta \\
&\Rightarrow -\frac{\varepsilon}{2} < x-1 < \frac{\varepsilon}{2} \\
&\Rightarrow -\varepsilon < 2x-2 < \varepsilon \\
&\Rightarrow -\varepsilon < 2x-2 < \varepsilon \\
&\Rightarrow |2x-2| < \varepsilon \\
&\Rightarrow |(2x+1)-3| < \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \to 1}(2x+1) = 3.$ (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0.$

Choose $\delta = \varepsilon/2.$

Assume $0 < |x-1| < \delta.$

Thus,

$$\begin{aligned}
|(2x+1)-3| \; &= |2x-2| \\
&= |2(x-1)| \\
&= |2||x-1| \\
&= 2|x-1| \\
&< 2 \cdot \delta \\
&= 2 \cdot \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \to 1}(2x+1) = 3.$

The following Problem-Solving Strategy summarizes the type of proof we worked out in **Example 2.39**.

Problem-Solving Strategy: Proving That $\lim_{x \to a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0.$
2. Next, we need to obtain a value for $\delta.$ After we have obtained this value, we make the following statement, filling in the blank with our choice of δ: Choose $\delta =$ _______.
3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x-a| < \delta.$
4. Next, based on this assumption, we need to show that $|f(x)-L| < \varepsilon,$ where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta.$
5. We conclude our proof with the statement: Therefore, $\lim_{x \to a} f(x) = L.$

Example 2.40

Proving a Statement about a Limit

Complete the proof that $\lim_{x \to -1}(4x + 1) = -3$ by filling in the blanks.

Let ______.

Choose $\delta = \_\_\_\_\_\_\_\_$.

Assume $0 < |x - \_\_\_\_\_\_\_\_| < \delta$.

Thus, $|\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\varepsilon$.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose $\delta = \_\_\_\_\_\_\_\_$.

Assume $0 < |x - (-1)| < \delta$. (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta\_\_\_\_\_\_\_\_\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x - (-1)| < \delta$ (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

2.27 Complete the proof that $\lim_{x \to 2}(3x - 2) = 4$ by filling in the blanks.

Let ________.

Choose $\delta = \_\_\_\_\_\_\_\_$.

Assume $0 < |x - \_\_\_\_| < \_\_\_\_$.

Thus,

$|\_\_\_\_\_\_\_\_ - \_\_\_\_| = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\varepsilon$.

Therefore, $\lim_{x \to 2}(3x - 2) = 4$.

In **Example 2.39** and **Example 2.40**, the proofs were fairly straightforward, since the functions with which we were working were linear. In **Example 2.41**, we see how to modify the proof to accommodate a nonlinear function.

Example 2.41

Proving a Statement about the Limit of a Specific Function (Geometric Approach)

Prove that $\lim_{x \to 2} x^2 = 4$.

Solution

1. Let $\varepsilon > 0$. The first part of the definition begins "For every $\varepsilon > 0$," so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.
2. Without loss of generality, assume $\varepsilon \le 4$. Two questions present themselves: Why do we want $\varepsilon \le 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for δ, we will discover that δ involves the quantity $\sqrt{4-\varepsilon}$. Consequently, we need $\varepsilon \le 4$. In answer to the second question: If we can find $\delta > 0$ that "works" for $\varepsilon \le 4$, then it will "work" for any $\varepsilon > 4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε, it is never okay to put a lower bound (other than zero) on ε.
3. Choose $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$. **Figure 2.41** shows how we made this choice of δ.

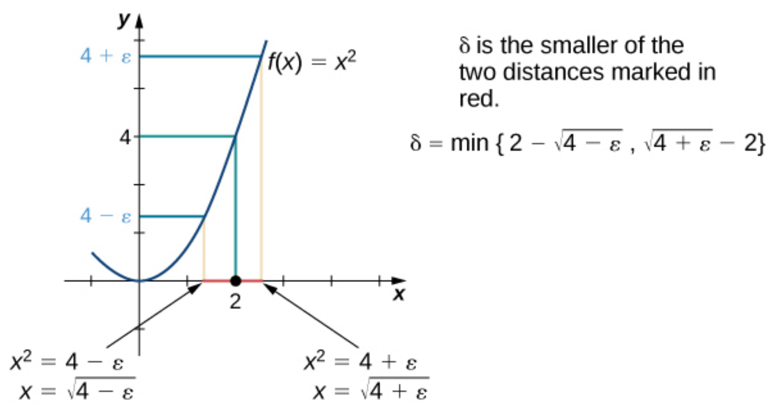


Figure 2.41 This graph shows how we find δ geometrically for a given ε for the proof in **Example 2.41**.

4. We must show: If $0 < |x-2| < \delta$, then $|x^2 - 4| < \varepsilon$, so we must begin by assuming

$$0 < |x-2| < \delta.$$

We don't really need $0 < |x-2|$ (in other words, $x \ne 2$) for this proof. Since $0 < |x-2| < \delta \Rightarrow |x-2| < \delta$, it is okay to drop $0 < |x-2|$.

$$|x-2| < \delta.$$

Hence,

$$-\delta < x - 2 < \delta.$$

Recall that $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$. Thus, $\delta \ge 2 - \sqrt{4-\varepsilon}$ and consequently $-(2 - \sqrt{4-\varepsilon}) \le -\delta$. We also use $\delta \le \sqrt{4+\varepsilon} - 2$ here. We might ask at this point: Why did we substitute $2 - \sqrt{4-\varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4+\varepsilon} - 2$ on the right-hand side of the inequality? If we look at **Figure 2.41**, we see that $2 - \sqrt{4-\varepsilon}$ corresponds to the distance on the left of 2 on the x-axis and $\sqrt{4+\varepsilon}-2$ corresponds to the distance on the right. Thus,

$$-\left(2-\sqrt{4-\varepsilon}\right) \leq -\delta < x-2 < \delta \leq \sqrt{4+\varepsilon}-2.$$

We simplify the expression on the left:

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2.$$

Then, we add 2 to all parts of the inequality:

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

$$4-\varepsilon < x^2 < 4+\varepsilon.$$

We subtract 4 from all parts of the inequality:

$$-\varepsilon < x^2-4 < \varepsilon.$$

Last,

$$|x^2-4| < \varepsilon.$$

5. Therefore,

$$\lim_{x \to 2} x^2 = 4.$$

2.28 Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x \to 9} \sqrt{x} = 3.$

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For **Example 2.42**, we take on a purely algebraic approach.

Example 2.42

Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x \to -1} \left(x^2-2x+3\right) = 6.$

Solution

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0$.
2. Choose $\delta = \min\{1, \varepsilon/5\}$. This choice of δ may appear odd at first glance, but it was obtained by

taking a look at our ultimate desired inequality: $\left|\left(x^2 - 2x + 3\right) - 6\right| < \varepsilon$. This inequality is equivalent to $|x + 1| \cdot |x - 3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \frac{\varepsilon}{x - 3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x - 3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \le 1$ comes into play. The choice of $\delta \le 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \le 1$ and $|x + 1| < \delta \le 1$, we are able to show that $|x - 3| < 5$. Consequently, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. At this point we realize that we also need $\delta \le \varepsilon/5$. Thus, we choose $\delta = \min\{1, \varepsilon/5\}$.

3. Assume $0 < |x + 1| < \delta$. Thus,

$$|x + 1| < 1 \text{ and } |x + 1| < \frac{\varepsilon}{5}.$$

Since $|x + 1| < 1$, we may conclude that $-1 < x + 1 < 1$. Thus, by subtracting 4 from all parts of the inequality, we obtain $-5 < x - 3 < -1$. Consequently, $|x - 3| < 5$. This gives us

$$\left|\left(x^2 - 2x + 3\right) - 6\right| = |x + 1| \cdot |x - 3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

$$\lim_{x \to -1} \left(x^2 - 2x + 3\right) = 6.$$

2.29 Complete the proof that $\lim_{x \to 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min\{1, \varepsilon/3\}$; assume $0 < |x - 1| < \delta$.

Since $|x - 1| < 1$, we may conclude that $-1 < x - 1 < 1$. Thus, $1 < x + 1 < 3$. Hence, $|x + 1| < 3$.

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review this key property of absolute value.

Definition

The **triangle inequality** states that if a and b are any real numbers, then $|a + b| \le |a| + |b|$.

Proof

We prove the following limit law: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} (f(x) + g(x)) = L + M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

Assume $0 < |x - a| < \delta$.

Thus,

$$0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2.$$

Hence,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

We now explore what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if there is no real number L for which $\lim_{x \to a} f(x) = L$. Thus, for all real numbers L, $\lim_{x \to a} f(x) \neq L$. To understand what this means, we look at each part of the definition of $\lim_{x \to a} f(x) = L$ together with its opposite. A translation of the definition is given in **Table 2.10**.

Definition	**Opposite**
1. For every $\varepsilon > 0$,	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0$, so that	2. for every $\delta > 0$,
3. if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.	3. There is an x satisfying $0 < |x - a| < \delta$ so that $|f(x) - L| \geq \varepsilon$.

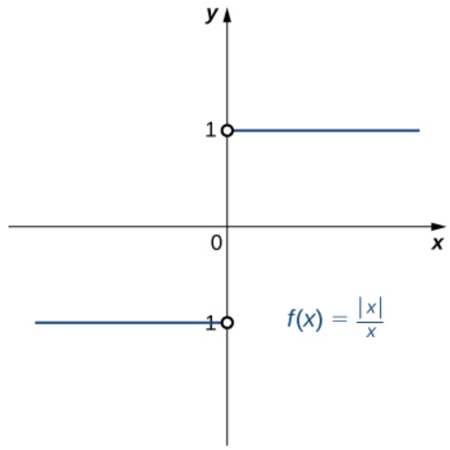
Table 2.10 Translation of the Definition of $\lim_{x \to a} f(x) = L$ and its Opposite

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x \to a} f(x)$ does not exist if for every real number L, there exists a real number $\varepsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < |x - a| < \delta$, so that $|f(x) - L| \geq \varepsilon$. Let's apply this in **Example 2.43** to show that a limit does not exist.

Example 2.43

Showing That a Limit Does Not Exist

Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$. Thus,

$$|x - 0| = \left|-\frac{\delta}{2} - 0\right| = \frac{\delta}{2} < \delta$$

and

$$\left|\frac{\left|-\frac{\delta}{2}\right|}{-\frac{\delta}{2}} - L\right| = |-1 - L| = L + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

$$|x - 0| = \left|\frac{\delta}{2} - 0\right| = \frac{\delta}{2} < \delta$$

and

$$\left|\frac{\left|\frac{\delta}{2}\right|}{\frac{\delta}{2}} - L\right| = |1 - L| = |L| + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

Thus, for any value of L, $\lim\limits_{x \to 0} \frac{|x|}{x} \neq L$.

One-Sided and Infinite Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a. Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a.

Definition

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a, b) where $a < b$. Then,

$$\lim_{x \to a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b, c) where $b < c$. Then,

$$\lim_{x \to a^-} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \varepsilon$.

Example 2.44

Proving a Statement about a Limit From the Right

Prove that $\lim_{x \to 4^+} \sqrt{x - 4} = 0$.

Solution

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $\left|\sqrt{x - 4} - 0\right| < \varepsilon$, we manipulate this inequality to get $\sqrt{x - 4} < \varepsilon$ or, equivalently, $0 < x - 4 < \varepsilon^2$, making $\delta = \varepsilon^2$ a clear choice. We may also determine δ geometrically, as shown in Figure 2.42.

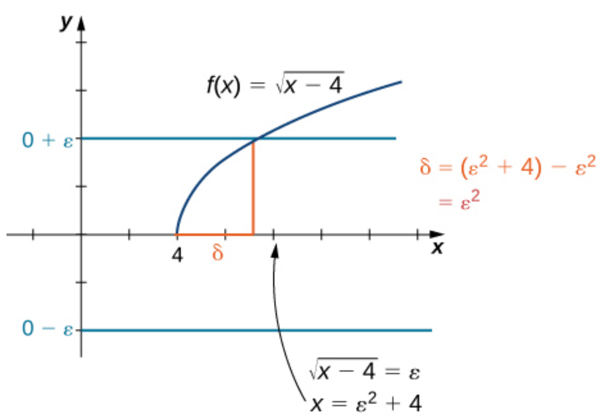


Figure 2.42 This graph shows how we find δ for the proof in **Example 2.44**.

Assume $0 < x - 4 < \delta$. Thus, $0 < x - 4 < \varepsilon^2$. Hence, $0 < \sqrt{x - 4} < \varepsilon$. Finally, $\left|\sqrt{x - 4} - 0\right| < \varepsilon$.

Therefore, $\lim_{x \to 4^+} \sqrt{x - 4} = 0$.

2.30 Find δ corresponding to ε for a proof that $\lim_{x \to 1^-} \sqrt{1 - x} = 0$.

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim_{x \to a} f(x) = +\infty,$ we want the values of the function $f(x)$ to get larger and larger as x approaches a. Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough $\delta,$ we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough $\delta.$ **Figure 2.43** illustrates this idea by showing the value of δ for successively larger values of M.

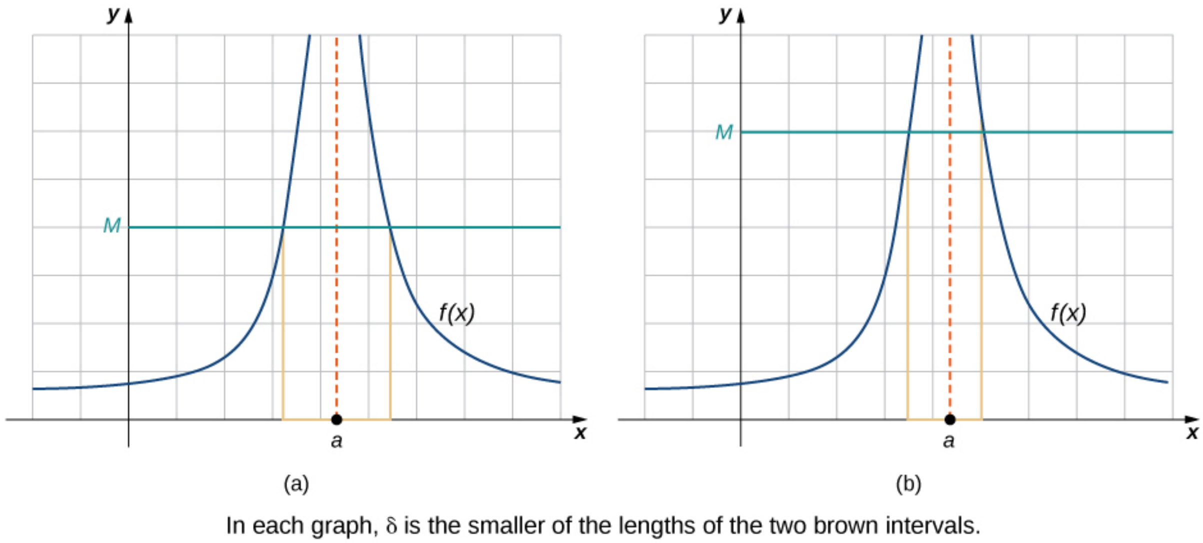


Figure 2.43 These graphs plot values of δ for M to show that $\lim_{x \to a} f(x) = +\infty.$

Definition

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have an infinite limit

$$\lim_{x \to a} f(x) = +\infty$$

if for every $M > 0,$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta,$ then $f(x) > M.$

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Then, we have a negative infinite limit

$$\lim_{x \to a} f(x) = -\infty$$

if for every $M > 0,$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta,$ then $f(x) < -M.$

Section 2.5: Continuity

This video provides an intuitive explanation of how to show continuity on a closed interval: Angie's Mathematical Insights - Continuity Closed Interval

This video provides an intuitive explanation of how to show continuity of trigonometric functions: Eric Hutchinson - Finding Continuity of Trigonometric Functions

2.4 | Continuity

Learning Objectives

2.4.1 Explain the three conditions for continuity at a point.
2.4.2 Describe three kinds of discontinuities.
2.4.3 Define continuity on an interval.
2.4.4 State the theorem for limits of composite functions.
2.4.5 Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity at a point* where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in **Figure 2.32**. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

$$\text{i. } f(a) \text{ is defined}$$

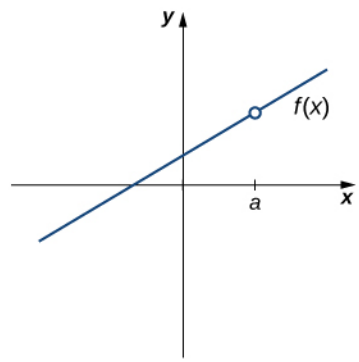


Figure 2.32 The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in **Figure 2.33**, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim\limits_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

$$\text{ii. } \lim_{x \to a} f(x) \text{ exists.}$$

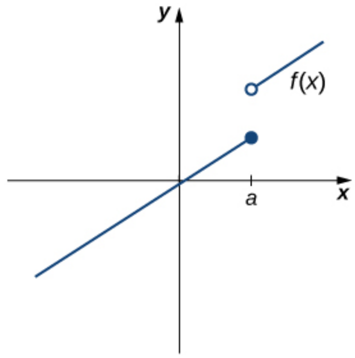


Figure 2.33 The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in **Figure 2.34**, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

$$\text{iii. } \lim_{x \to a} f(x) = f(a).$$

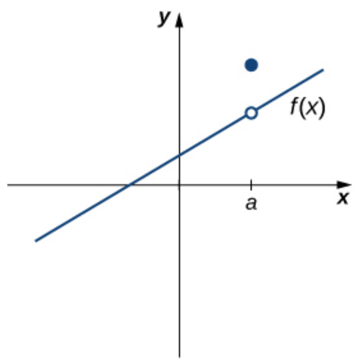


Figure 2.34 The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.26

Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = (x^2 - 4)/(x - 2)$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.35.

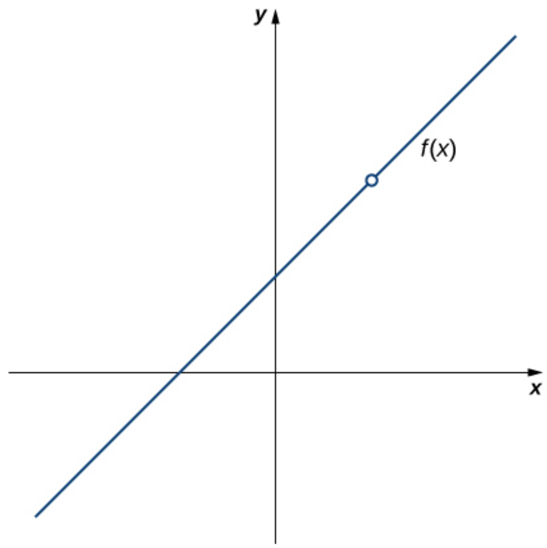


Figure 2.35 The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.27

Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \to 3} f(x)$. To do this, we must compute $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in **Figure 2.36**.

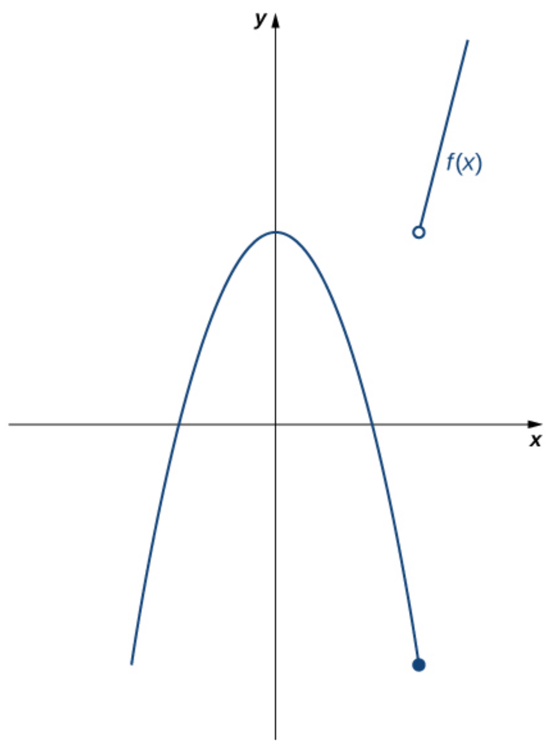


Figure 2.36 The function $f(x)$ is not continuous at 3 because $\lim_{x \to 3} f(x)$ does not exist.

Example 2.28

Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution

First, observe that

$$f(0) = 1.$$

Next,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \to 1} f(x)$. We see that

$$f(0) = 1 = \lim_{x \to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

2.21 Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.8: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply **Continuity of Polynomials and Rational Functions** to determine the points at which a given rational function is continuous.

Example 2.29

Continuity of a Rational Function

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

2.22 For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Types of Discontinuities

As we have seen in Example 2.26 and Example 2.27, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.37 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

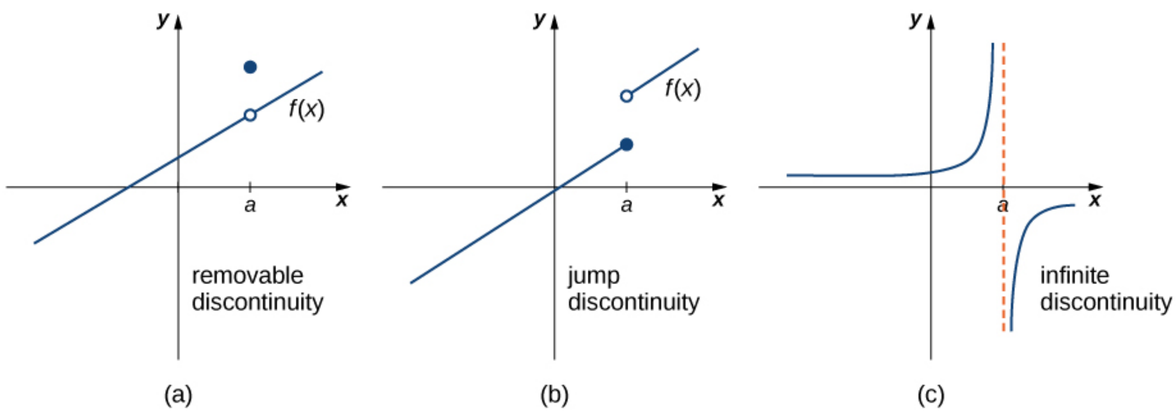


Figure 2.37 Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Example 2.30

Classifying a Discontinuity

In **Example 2.26**, we showed that $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim_{x \to 2} f(x)$:

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \to 2} (x + 2) \\ &= 4. \end{aligned}$$

Since *f* is discontinuous at 2 and $\lim_{x \to 2} f(x)$ exists, *f* has a removable discontinuity at $x = 2$.

Example 2.31

Classifying a Discontinuity

In **Example 2.27**, we showed that $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that *f* is discontinuous at 3 because $\lim_{x \to 3} f(x)$ does not exist. However, since $\lim_{x \to 3^-} f(x) = -5$ and $\lim_{x \to 3^-} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.32

Classifying a Discontinuity

Determine whether $f(x) = \frac{x + 2}{x + 1}$ is continuous at −1. If the function is discontinuous at −1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at −1. To determine the type of discontinuity, we must determine the limit at −1. We see that $\lim_{x \to -1^-} \frac{x+2}{x+1} = -\infty$ and $\lim_{x \to -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at −1.

2.23 For $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether *f* is continuous at 1. If *f* is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at *a* if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at *a* if $\lim_{x \to a^-} f(x) = f(a)$.

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at *a* and is continuous from the left at *b*. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at *b*. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 2.33

Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals

$$(-\infty, -2), (-2, 0), \text{ and } (0, +\infty).$$

Example 2.34

Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim\limits_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim\limits_{x \to -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim\limits_{x \to 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

2.24 State the interval(s) over which the function $f(x) = \sqrt{x + 3}$ is continuous.

The **Composite Function Theorem** allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 2.9: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim\limits_{x \to a} g(x) = L$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

Before we move on to **Example 2.35**, recall that earlier, in the section on limit laws, we showed $\lim\limits_{x \to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In **Example 2.35** we see how to combine this result with the composite function theorem.

Example 2.35

Limit of a Composite Cosine Function

Evaluate $\lim\limits_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x \to \pi/2}\left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x \to \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \to \pi/2}\left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

2.25 Evaluate $\lim_{x \to \pi} \sin(x - \pi)$.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.10: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x \to a} \cos x = \cos a$ for all values of a.

$\lim_{x \to a} \cos x$	$= \lim_{x \to a} \cos((x - a) + a)$	rewrite $x = x - a + a$
	$= \lim_{x \to a}(\cos(x - a)\cos a - \sin(x - a)\sin a)$	apply the identity for the cosine of the sum of two angles
	$= \cos\left(\lim_{x \to a}(x - a)\right)\cos a - \sin\left(\lim_{x \to a}(x - a)\right)\sin a$	$\lim_{x \to a}(x - a) = 0,$ and $\sin x$ and $\cos x$ are continuous at 0
	$= \cos(0)\cos a - \sin(0)\sin a$	evaluate cos(0) and sin(0) and simplify
	$= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.$	

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x,$ their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b],$ where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Theorem 2.11: The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b),$ then there is a number c in $[a, b]$ satisfying $f(c) = z$ in **Figure 2.38**.

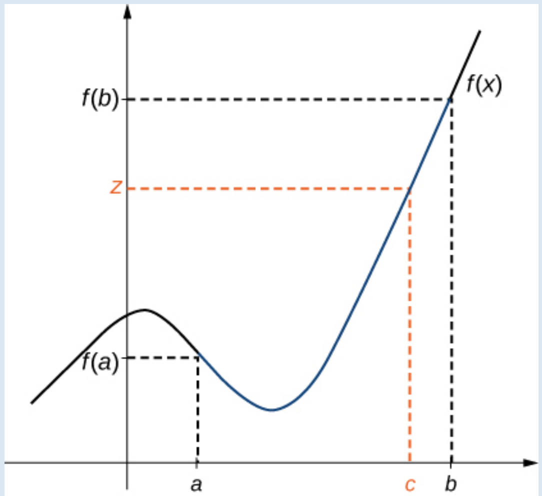


Figure 2.38 There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Example 2.36

Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 2.37

When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x - 1)^2$. It satisfies $f(0) = 1 > 0$, $f(2) = 1 > 0$, and $f(1) = 0$.

Example 2.38

When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

2.26 Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Section 2.6: Limits at Infinity; Horizontal Asymptotes

This video provides an intuitive explanation of infinite limits: Khan Academy - Infinite Limits intro

This video provides an explanation of how to find a limit at infinity: The Organic Chemistry Tutor - How to Find the Limit at Infinity

This video provides an example of a proof of an infinite limit: Dr Peyam - Epsilon delta limit (Example 4): Limits at Infinity

3.5 Infinite Limits and Limits at Infinity

We occasionally want to know what happens to some quantity when a variable gets very large or "goes to infinity".

Example 3.15: Limit at Infinity

What happens to the function $\cos(1/x)$ *as* x *goes to infinity? It seems clear that as* x *gets larger and larger,* $1/x$ *gets closer and closer to zero, so* $\cos(1/x)$ *should be getting closer and closer to* $\cos(0) = 1$.

As with ordinary limits, this concept of "limit at infinity" can be made precise. Roughly, we want $\lim_{x\to\infty} f(x) = L$ to mean that we can make $f(x)$ as close as we want to L by making x large enough.

Source: Lyryx Learning; Calculus: Early Transcendentals; 2016 , pg 86

Definition 3.16: Limit at Infinity (Formal Definition)

If f is a function, we say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is an $N > 0$ so that whenever $x > N$, $|f(x) - L| < \varepsilon$. We may similarly define $\lim_{x\to-\infty} f(x) = L$.

We include this definition for completeness, but we will not explore it in detail. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there is a direct analog of Theorem 3.8.

Example 3.17: Limit at Infinity

Compute $\lim_{x\to\infty} \dfrac{2x^2 - 3x + 7}{x^2 + 47x + 1}$.

Solution. As x goes to infinity both the numerator and denominator go to infinity. We divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x\to\infty} \frac{2 - \dfrac{3}{x} + \dfrac{7}{x^2}}{1 + \dfrac{47}{x} + \dfrac{1}{x^2}}.$$

Now as x approaches infinity, all the quotients with some power of x in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2. ♣

In the previous example, we *divided by the highest power of x that occurs in the denominator* in order to evaluate the limit. We illustrate another technique similar to this.

Example 3.18: Limit at Infinity

Compute the following limit:

$$\lim_{x\to\infty} \frac{2x^2 + 3}{5x^2 + x}.$$

Solution. As x becomes large, both the numerator and denominator become large, so it isn't clear what happens to their ratio. The highest power of x in the denominator is x^2, therefore we will divide every term in both the numerator and denominator by x^2 as follows:

$$\lim_{x\to\infty} \frac{2x^2 + 3}{5x^2 + x} = \lim_{x\to\infty} \frac{2 + 3/x^2}{5 + 1/x}.$$

Most of the limit rules from last lecture also apply to infinite limits, so we can write this as:

$$= \frac{\lim_{x\to\infty} 2 + 3 \lim_{x\to\infty} \dfrac{1}{x^2}}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \dfrac{1}{x}} = \frac{2 + 3(0)}{5 + 0} = \frac{2}{5}.$$

Note that we used the theorem above to get that $\lim_{x\to\infty} \dfrac{1}{x} = 0$ and $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$.

A shortcut technique is to analyze only the *leading terms* of the numerator and denominator. A leading term is a term that has the highest power of x. If there are multiple terms with the same exponent, you must include all of them.
Top: The leading term is $2x^2$.
Bottom: The leading term is $5x^2$.
Now only looking at leading terms and ignoring the other terms we get:

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x} = \lim_{x\to\infty} \frac{2x^2}{5x^2} = \frac{2}{5}.$$

♣

We next look at limits whose value is infinity (or minus infinity).

Definition 3.19: Infinite Limit (Useable Definition)

In general, we will write

$$\lim_{x\to a} f(x) = \infty$$

if we can make the value of $f(x)$ arbitrarily large by taking x to be sufficiently close to a (on either side of a) but not equal to a. Similarly, we will write

$$\lim_{x\to a} f(x) = -\infty$$

if we can make the value of $f(x)$ arbitrarily large and negative by taking x to be sufficiently close to a (on either side of a) but not equal to a.

This definition can be modified for one-sided limits as well as limits with $x \to a$ replaced by $x \to \infty$ or $x \to -\infty$.

Example 3.20: Limit at Infinity

Compute the following limit: $\lim_{x\to\infty} (x^3 - x)$.

Solution. One might be tempted to write:

$$\lim_{x\to\infty} x^3 - \lim_{x\to\infty} x = \infty - \infty,$$

however, we do not know what $\infty - \infty$ is, as ∞ is not a real number and so cannot be treated like one. We instead write:

$$\lim_{x\to\infty} (x^3 - x) = \lim_{x\to\infty} x(x^2-1).$$

As x becomes arbitrarily large, then both x and $x^2 - 1$ become arbitrarily large, and hence their product $x(x^2-1)$ will also become arbitrarily large. Thus we see that

$$\lim_{x\to\infty} (x^3 - x) = \infty.$$

♣

Example 3.21: Limit at Infinity and Basic Functions

We can easily evaluate the following limits by observation:

1. $\lim_{x\to\infty} \frac{6}{\sqrt{x^3}} = 0$
2. $\lim_{x\to-\infty} x - x^2 = -\infty$
3. $\lim_{x\to\infty} x^3 + x = \infty$
4. $\lim_{x\to\infty} \cos(x) = DNE$
5. $\lim_{x\to\infty} e^x = \infty$
6. $\lim_{x\to-\infty} e^x = 0$
7. $\lim_{x\to 0^+} \ln x = -\infty$
8. $\lim_{x\to 0} \cos(1/x) = DNE$

Often, the shorthand notation $\frac{1}{0^+} = +\infty$ and $\frac{1}{0^-} = -\infty$ is used to represent the following two limits respectively:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Using the above convention we can compute the following limits.

Example 3.22: Limit at Infinity and Basic Functions

Compute $\lim_{x\to 0^+} e^{1/x}$, $\lim_{x\to 0^-} e^{1/x}$ *and* $\lim_{x\to 0} e^{1/x}$.

Solution. We have:

$$\lim_{x\to 0^+} e^{\frac{1}{x}} = e^{\frac{1}{0^+}} = e^{+\infty} = \infty.$$

$$\lim_{x\to 0^-} e^{\frac{1}{x}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0.$$

Thus, as left-hand limit $\neq$ right-hand limit,

$$\lim_{x\to 0} e^{\frac{1}{x}} = \text{DNE}.$$

♣

3.5.1. Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of $f(x)$ if *at least one* of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example 3.23: Vertical Asymptotes

Find the vertical asymptotes of $f(x) = \dfrac{2x}{x-4}$.

Solution. In the definition of vertical asymptotes we need a certain limit to be $\pm\infty$. Candidates would be to consider values not in the domain of $f(x)$, such as $a = 4$. As x approaches 4 but is larger than 4 then $x-4$ is a small positive number and $2x$ is close to 8, so the quotient $2x/(x-4)$ is a large positive number. Thus we see that

$$\lim_{x\to 4^+} \frac{2x}{x-4} = \infty.$$

Thus, at least one of the conditions in the definition above is satisfied. Therefore $x = 4$ is a vertical asymptote. ♣

3.5.2. Horizontal Asymptotes

The line $y = L$ is a **horizontal asymptote** of $f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Example 3.24: Horizontal Asymptotes

Find the horizontal asymptotes of $f(x) = \dfrac{|x|}{x}$.

Solution. We must compute two infinite limits. First,

$$\lim_{x\to\infty} \frac{|x|}{x}.$$

Notice that for x arbitrarily large that $x > 0$, so that $|x| = x$. In particular, for x in the interval $(0, \infty)$ we have

$$\lim_{x\to\infty} \frac{|x|}{x} = \lim_{x\to\infty} \frac{x}{x} = 1.$$

Second, we must compute

$$\lim_{x\to-\infty} \frac{|x|}{x}.$$

Notice that for x arbitrarily large negative that $x < 0$, so that $|x| = -x$. In particular, for x in the interval $(-\infty, 0)$ we have

$$\lim_{x\to-\infty} \frac{|x|}{x} = \lim_{x\to-\infty} \frac{-x}{x} = -1.$$

Therefore there are two horizontal asymptotes, namely, $y = 1$ and $y = -1$. ♣

3.5.3. Slant Asymptotes

Some functions may have slant (or *oblique*) asymptotes, which are neither vertical nor horizontal. If

$$\lim_{x\to\infty}[f(x)-(mx+b)]=0$$

then the straight line $y=mx+b$ is a **slant asymptote** to $f(x)$. Visually, the vertical distance between $f(x)$ and $y=mx+b$ is decreasing towards 0 and the curves do not intersect or cross at any point as x approaches infinity. Similarly when $x\to-\infty$.

Example 3.25: Slant Asymptote in a Rational Function

Find the slant asymptotes of $f(x)=\dfrac{-3x^2+4}{x-1}$.

Solution. Note that this function has no horizontal asymptotes since $f(x)\to-\infty$ as $x\to\infty$ and $f(x)\to\infty$ as $x\to-\infty$.

In rational functions, slant asymptotes occur when the degree in the numerator is one greater than in the denominator. We use long division to rearrange the function:

$$\frac{-3x^2+4}{x-1}=-3x-3+\frac{1}{x-1}.$$

The part we're interested in is the resulting polynomial $-3x-3$. This is the line $y=mx+b$ we were seeking, where $m=-3$ and $b=-3$. Notice that

$$\lim_{x\to\infty}\frac{-3x^2+4}{x-1}-(-3x-3)=\lim_{x\to\infty}\frac{1}{x-1}=0$$

and

$$\lim_{x\to-\infty}\frac{-3x^2+4}{x-1}-(-3x-3)=\lim_{x\to-\infty}\frac{1}{x-1}=0.$$

Thus, $y=-3x-3$ is a slant asymptote of $f(x)$. ♣

Although rational functions are the most common type of function we encounter with slant asymptotes, there are other types of functions we can consider that present an interesting challenge.

Example 3.26: Slant Asymptote

Show that $y=2x+4$ *is a slant asymptote of* $f(x)=2x-3^x+4$.

Solution. This is because

$$\lim_{x\to-\infty}[f(x)-(2x+4)]=\lim_{x\to-\infty}(-3^x)=0.$$

♣

We note that $\lim_{x\to\infty}[f(x)-(2x+4)]=\lim_{x\to\infty}(-3^x)=-\infty$. So, the vertical distance between $y=f(x)$ and the line $y=2x+4$ decreases toward 0 only when $x\to-\infty$ and not when $x\to\infty$. The graph of f

approaches the slant asymptote $y = 2x+4$ only at the far left and not at the far right. One might ask if $y = f(x)$ approaches a slant asymptote when $x \to \infty$. The answer turns out to be no, but we will need to know something about the relative growth rates of the exponential functions and linear functions in order to prove this. Specifically, one can prove that when the base is greater than 1 the exponential functions grows faster than any power function as $x \to \infty$. This can be phrased like this: For any $a > 1$ and any $n > 0$,

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \infty \text{ and } \lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

These facts are most easily proved with the aim of something called the L'Hôpital's Rule.

3.5.4. End Behaviour and Comparative Growth Rates

Let us now look at the last two subsections and go deeper. In the last two subsections we looked at horizontal and slant asymptotes. Both are special cases of the end behaviour of functions, and both concern situations where the graph of a function approaches a straight line as $x \to \infty$ or $-\infty$. But not all functions have this kind of end behviour. For example, $f(x) = x^2$ and $f(x) = x^3$ do not approach a straight line as $x \to \infty$ or $-\infty$. The best we can say with the notion of limit developed at this stage are that

$$\begin{aligned} \lim_{x\to\infty} x^2 &= \infty, \ \lim_{x\to-\infty} x^2 = \infty, \\ \lim_{x\to\infty} x^3 &= \infty, \ \lim_{x\to-\infty} x^3 = -\infty. \end{aligned}$$

Similarly, we can describe the end behaviour of transcendental functions such as $f(x) = e^x$ using limits, and in this case, the graph approaches a line as $x \to -\infty$ but not as $x \to \infty$.

$$\lim_{x\to-\infty} e^x = 0, \ \lim_{x\to\infty} e^x = \infty.$$

People have found it useful to make a finer distinction between these end behaviours all thus far captured by the symbols ∞ and $-\infty$. Specifically, we will see that the above functions have different growth rates at infinity. Some increases to infinty faster than others. Specifically,

Definition 3.27: Comparative Growth Rates

Suppose that f and g are two functions such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. We say that $f(x)$ grows faster than $g(x)$ as $x \to \infty$ if the following holds:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

or equivalently,

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

Here are a few obvious examples:

Example 3.28:

Show that if $m > n$ are two positive integers, then $f(x) = x^m$ grows faster than $g(x) = x^n$ as $x \to \infty$.

Solution. Since $m > n$, $m - n$ is a positive integer. Therefore,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty.$$

♣

Example 3.29:

Show that if $m > n$ are two positive integers, then any monic polynomial $P_m(x)$ of degree m grows faster than any monic polynomial $P_n(x)$ of degree n as $x \to \infty$. [Recall that a polynomial is monic if its leading coefficient is 1.]

Solution. By assumption, $P_m(x) = x^m+$ terms of degrees less than $m = x^m + a_{m-1}x^{m-1} + \ldots$, and $P_n(x) = x^n+$ terms of degrees less than $n = x^n + b_{n-1}x^{n-1} + \ldots$. Dividing the numerator and denominator by x^n, we get

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{m-n} + a_{m-1}x^{m-n-1} + \ldots}{1 + \frac{b_{n-1}}{x} + \ldots} = \lim_{x\to\infty} x^{m-n}\left(\frac{1 + \frac{a_{m-1}}{x} + \ldots}{1 + \frac{b_{n-1}}{x} + \ldots}\right) = \infty,$$

since the limit of the bracketed fraction is 1 and the limit of x^{m-n} is ∞, as we showed in Example 3.28.

♣

Example 3.30:

Show that a polynomial grows exactly as fast as its highest degree term as $x \to \infty$ or $-\infty$. That is, if $P(x)$ is any polynomial and $Q(x)$ is its highest degree term, then both limits

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} \text{ and } \lim_{x\to-\infty} \frac{P(x)}{Q(x)}$$

are finite and nonzero.

Solution. Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$. Then the highest degree term is $Q(x) = a_n x^n$. So,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) = a_n \neq 0.$$

♣

Let's state a theorem we mentioned when we discussed the last example in the last subsection:

Theorem 3.31:

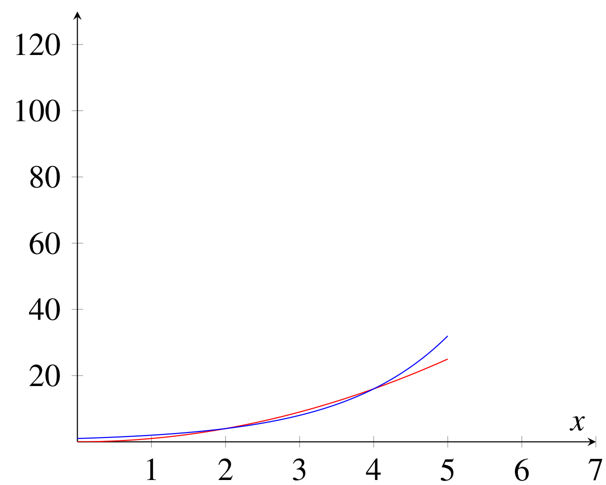
Let n be any positive integer and let $a > 1$. Then $f(x) = a^x$ grows faster than $g(x) = x^n$ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \infty, \ \lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

In particular,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty, \ \lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

The easiest way to prove this is to use the L'Hôpital's Rule, which we will introduce in a later chapter. For now, one can plot and compare the graphs of an exponential function and a power function. Here is a comparison between $f(x) = x^2$ and $g(x) = 2^x$:



Notice also that as $x \to -\infty$, x^n grows in size but e^x does not. More specifically, $x^n \to \infty$ or $-\infty$ according as n is even or odd, while $e^x \to 0$. So, it is meaningless to compare their "growth" rates, although we can still calculate the limit

$$\lim_{x\to-\infty} \frac{e^x}{x^n} = 0.$$

Let's see an application of our theorem.

Example 3.32:

Find the horizontal asymptote(s) of $f(x) = \dfrac{x^3 + 2e^x}{e^x - 4x^2}$.

Solution. To find horizontal asymptotes, we calculate the limits of $f(x)$ as $x \to \infty$ and $x \to -\infty$. For $x \to \infty$, we divide the numerator and the denominator by e^x, and then we take limit to get

$$\lim_{x\to\infty} \frac{x^3 + 2e^x}{e^x - 4x^2} = \lim_{x\to\infty} \frac{\frac{x^3}{e^x} + 2}{1 - 4\frac{x^2}{e^x}} = \frac{0+2}{1-4(0)} = 2.$$

For $x \to -\infty$, we divide the numerator and the denominator by x^2 to get

$$\lim_{x\to-\infty} \frac{x^3+2e^x}{e^x-4x^2} = \lim_{x\to-\infty} \frac{x+2\frac{e^x}{x^2}}{\frac{e^x}{x^2}-4}.$$

The denominator now approaches $0-4=-4$. The numerator has limit $-\infty$. So, the quotient has limit ∞:

$$\lim_{x\to-\infty} \frac{x+2\frac{e^x}{x^2}}{\frac{e^x}{x^2}-4} = \infty.$$

So, $y=2$ is a horizontal asymptote. The function $y=f(x)$ approaches the line $y=2$ as $x\to\infty$. And this is the only horizontal asymptote, since the function $y=f(x)$ does not approach any horizontal line as $x\to-\infty$. ♣

Since the growth rate of a polynomial is the same as that of its leading term, the following is obvious:

Example 3.33:

If $P(x)$ is any polynomial, then

$$\lim_{x\to\infty} \frac{P(x)}{e^x} = 0.$$

Also, if r is any real number, then we can place it between two consecutive integers n and $n+1$. For example, $\sqrt{3}$ is between 1 and 2, e is between 2 and 3, and π is between 3 and 4. Then the following is totally within our expectation:

Example 3.34:

Prove that if $a>1$ is any basis and $r>0$ is any exponent, then $f(x)=a^x$ grows faster than $g(x)=x^r$ as $x\to\infty$.

Solution. Let r be between consecutive integers n and $n+1$. Then for all $x>1$, $x^n \le x^r \le x^{n+1}$. Dividing by a^x, we get

$$\frac{x^n}{a^x} \le \frac{x^r}{a^x} \le \frac{x^{n+1}}{a^x}.$$

Since

$$\lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

♣

What about exponential functions with different bases? We recall from the graphs of the exponential functions that for any base $a>1$,

$$\lim_{x\to\infty} a^x = \infty.$$

So, the exponential functions with bases greater than 1 all grow to infinity as $x\to\infty$. How do their growth rates compare?

Theorem 3.35:

If $1 < a < b$, then $f(x) = b^x$ grows faster than $g(x) = a^x$ as $x \to \infty$.

Proof. Proof. Since $a < b$, we have $\frac{b}{a} > 1$. So,

$$\lim_{x\to\infty} \frac{b^x}{a^x} = \lim_{x\to\infty} \left(\frac{b}{a}\right)^x = \infty.$$

♣

Another function that grows to infinity as $x \to \infty$ is $g(x) = \ln x$. Recall that the natural logarithmic function is the inverse of the exponential function $y = e^x$. Since e^x grows very fast as x increases, we should expect $\ln x$ to grow very slowly as x increases. The same applies to logarithmic functions with any basis $a > 1$. This is the content of the next theorem.

Theorem 3.36:

Let r be any positive real number and $a > 1$. Then

(a) $f(x) = x^r$ grows faster than $g(x) = \ln x$ as $x \to \infty$.

(b) $f(x) = x^r$ grows faster than $g(x) = \log_a x$ as $x \to \infty$.

Proof.

1. We use a change of variable. Letting $t = \ln x$, then $x = e^t$. So, $x \to \infty$ if and only if $t \to \infty$, and

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{t\to\infty} \frac{t}{(e^t)^r} = \lim_{t\to\infty} \frac{t}{(e^r)^t}.$$

   Now, since $r > 0$, $a = e^r > 1$. So, a^t grows as t increases, and it grows faster than t as $t \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{t\to\infty} \frac{t}{(e^r)^t} = \lim_{t\to\infty} \frac{t}{a^t} = 0.$$

2. The change of base identity $\log_a x = \dfrac{\ln x}{\ln a}$ implies that $\log_a x$ is simply a constant multple of $\ln x$. The result now follows from (a).

Section 2.7: Derivatives and Rates of Change

well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

3.1 | Defining the Derivative

Learning Objectives

- **3.1.1** Recognize the meaning of the tangent to a curve at a point.
- **3.1.2** Calculate the slope of a tangent line.
- **3.1.3** Identify the derivative as the limit of a difference quotient.
- **3.1.4** Calculate the derivative of a given function at a point.
- **3.1.5** Describe the velocity as a rate of change.
- **3.1.6** Explain the difference between average velocity and instantaneous velocity.
- **3.1.7** Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton $(1643–1727)$ and the German Gottfried Leibniz $(1646–1716)$, whose images appear in **Figure 3.2**. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

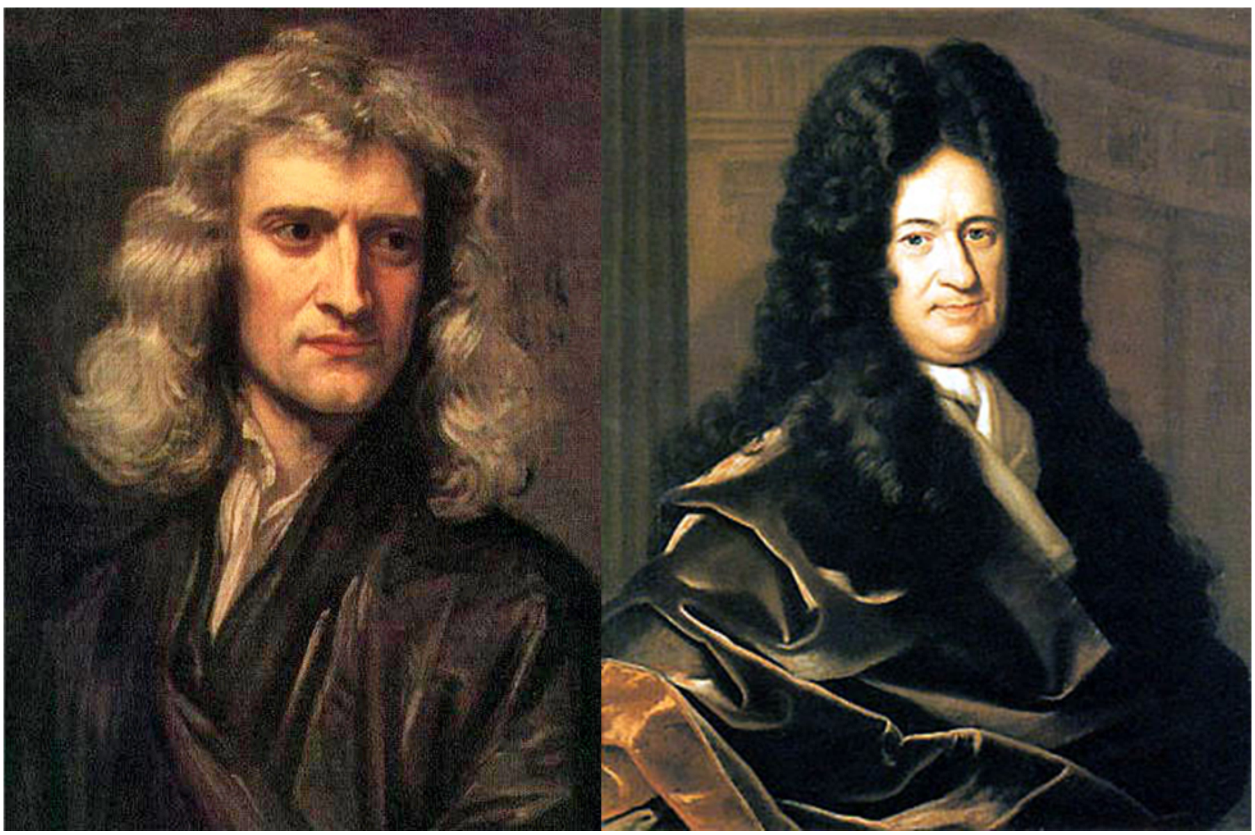


Figure 3.2 Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in **Figure 3.3**. The slope of this line is given by an equation in the form of a difference quotient:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to 0. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}.$$

Definition

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

$$Q = \frac{f(x) - f(a)}{x - a} \quad (3.1)$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

$$Q = \frac{f(a + h) - f(a)}{h} \quad (3.2)$$

is a difference quotient with increment h.

View the development of the **derivative (http://www.openstaxcollege.org/l/20_calcapplets)** with this applet.

These two expressions for calculating the slope of a secant line are illustrated in **Figure 3.3**. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.

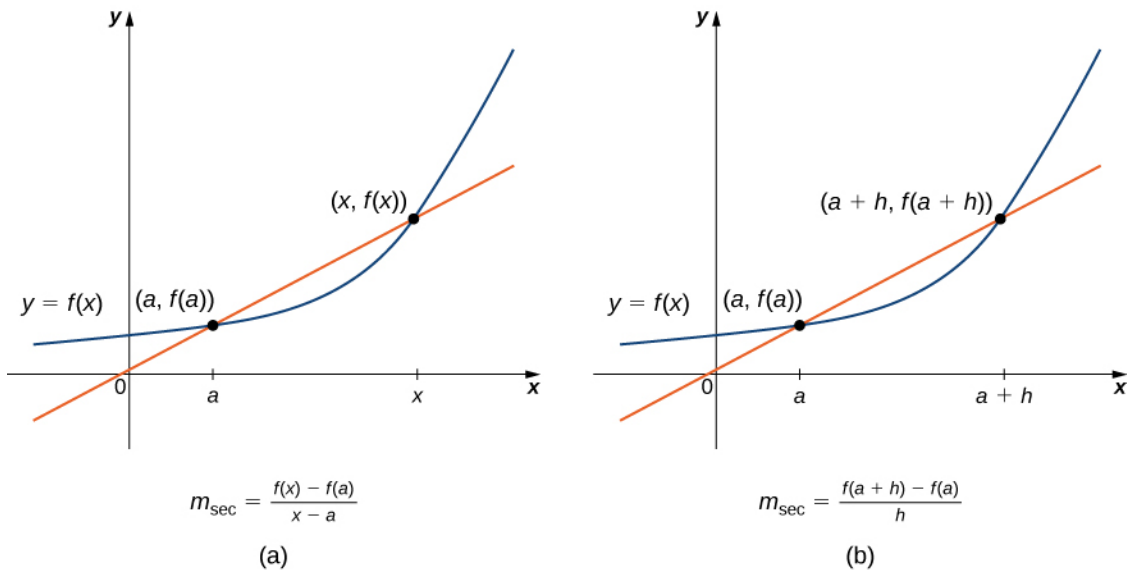


Figure 3.3 We can calculate the slope of a secant line in either of two ways.

In **Figure 3.4**(a) we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, **Figure 3.4**(b) shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in **Figure 3.4**(c).

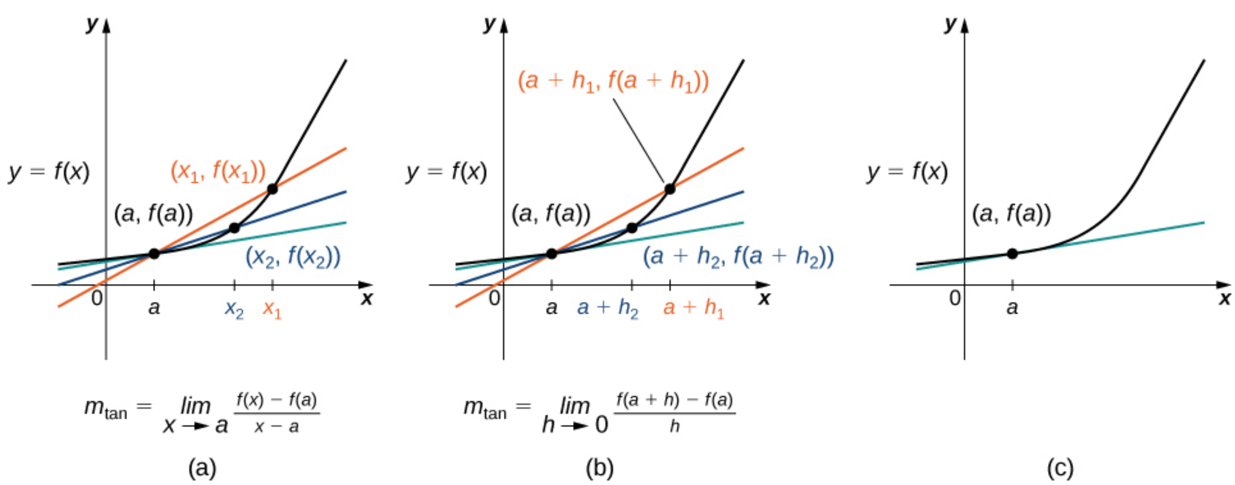


Figure 3.4 The secant lines approach the tangent line (shown in green) as the second point approaches the first.

You can use this **site (http://www.openstaxcollege.org/l/20_diffmicros)** to explore graphs to see if they have a tangent line at a point.

In **Figure 3.5** we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be locally linear in the immediate vicinity of $x = 1$.

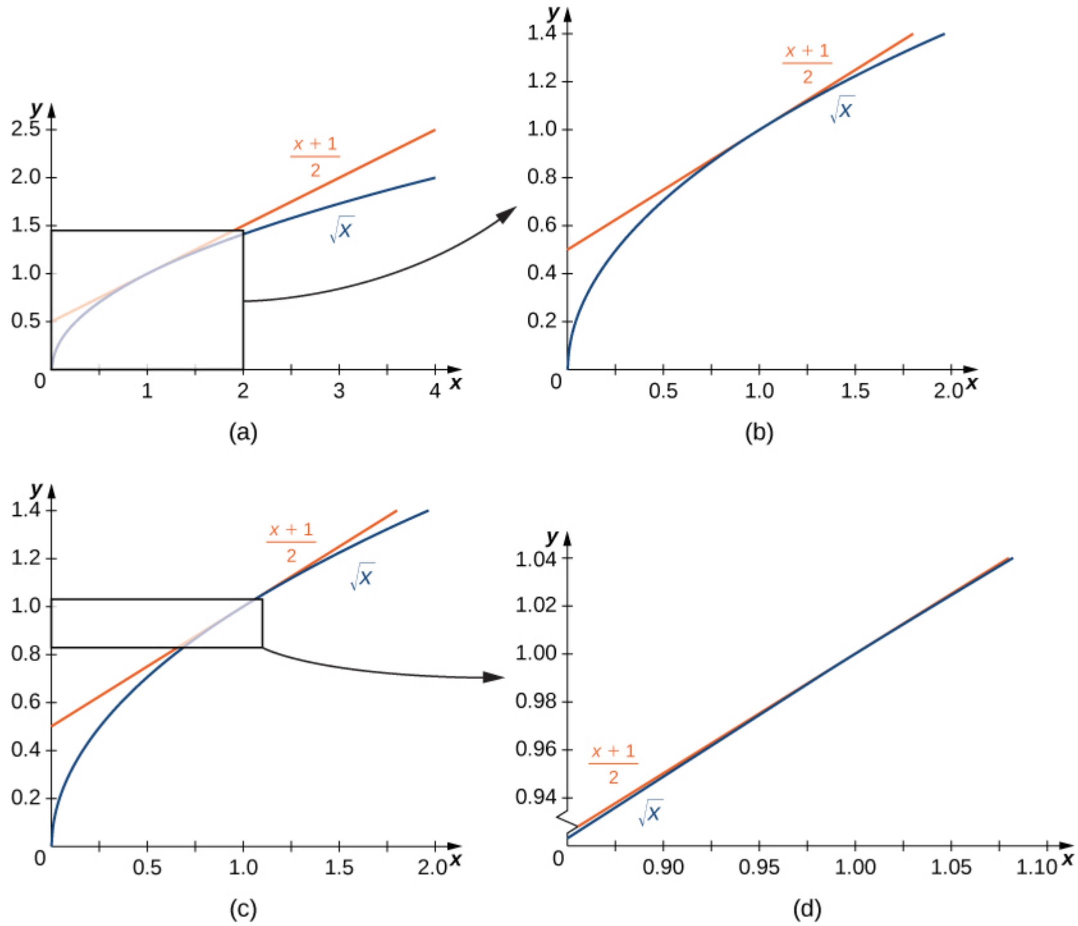


Figure 3.5 For values of x close to $1,$ the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.3}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{3.4}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example 3.1

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

First find the slope of the tangent line. In this example, use **Equation 3.3**.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} && \text{Apply the definition} \\ &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9. \\ &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3}(x + 3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in **Figure 3.6**.

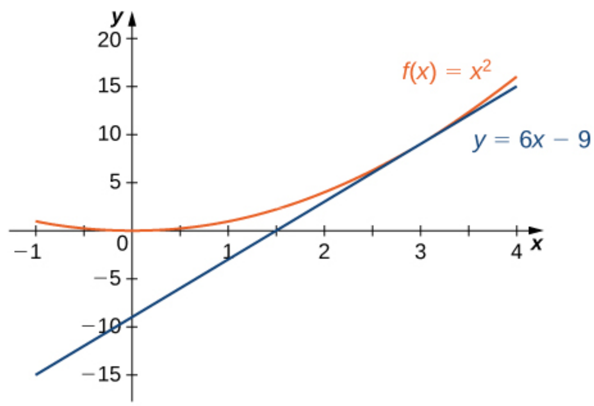


Figure 3.6 The tangent line to $f(x)$ at $x = 3$.

Example 3.2

The Slope of a Tangent Line Revisited

Use **Equation 3.4** to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution

The steps are very similar to **Example 3.1**. See **Equation 3.4** for the definition.

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} && \text{Apply the definition} \\ &= \lim_{h \to 0} \frac{(3+h)^2 - 9}{h} && \text{Substitute } f(3+h) = (3+h)^2 \text{ and } f(3) = 9. \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} && \text{Expand and simplify to evaluate the limit.} \\ &= \lim_{h \to 0} \frac{h(6+h)}{h} = \lim_{h \to 0} (6+h) = 6 \end{aligned}$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example 3.3

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution

We can use **Equation 3.3**, but as we have seen, the results are the same if we use **Equation 3.4**.

$$\begin{aligned} m_{\tan} &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition} \\ &= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} && \text{Substitute } f(x) = \frac{1}{x} \text{ and } f(2) = \frac{1}{2}. \\ &= \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \cdot \frac{2x}{2x} && \text{Multiply numerator and denominator by } 2x \text{ to simplify fractions.} \\ &= \lim_{x \to 2} \frac{(2 - x)}{(x - 2)(2x)} && \text{Simplify.} \\ &= \lim_{x \to 2} \frac{-1}{2x} && \text{Simplify using } \frac{2 - x}{x - 2} = -1, \text{ for } x \neq 2. \\ &= -\frac{1}{4} && \text{Evaluate the limit.} \end{aligned}$$

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in **Figure 3.7**.

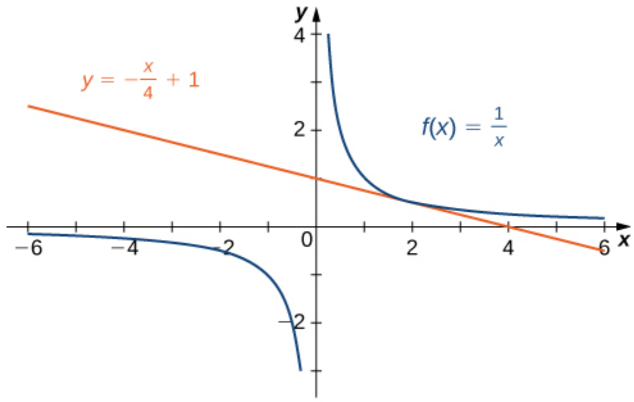


Figure 3.7 The line is tangent to $f(x)$ at $x = 2$.

3.1 Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative**. The process of finding a derivative is called **differentiation**.

Definition

Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{3.5}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}. \tag{3.6}$$

Example 3.4

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using **Equation 3.5**.

Solution

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

3.2 For $f(x) = x^2$, use a table to estimate $f'(3)$ using **Equation 3.6**.

Example 3.5

Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using **Equation 3.5**.

Solution

Substitute the given function and value directly into the equation.

$$\begin{aligned}
f'(x) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} && \text{Apply the definition} \\
&= \lim_{x \to 2} \frac{\left(3x^2 - 4x + 1\right) - 5}{x - 2} && \text{Substitute } f(x) = 3x^2 - 4x + 1 \text{ and } f(2) = 5. \\
&= \lim_{x \to 2} \frac{(x-2)(3x+2)}{x-2} && \text{Simplify and factor the numerator.} \\
&= \lim_{x \to 2} (3x + 2) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}$$

Example 3.6

Revisiting the Derivative

For $f(x) = 3x^2 - 4x + 1,$ find $f'(2)$ by using **Equation 3.6**.

Solution

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in **Example 3.5**.

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} && \text{Apply the definition} \\
&= \lim_{h \to 0} \frac{(3(2+h)^2 - 4(2+h) + 1) - 5}{h} && \text{Substitute } f(2) = 5 \text{ and } f(2+h) = 3(2+h)^2 - 4(2+h) + 1. \\
&= \lim_{h \to 0} \frac{3h^2 + 8h}{h} && \text{Simplify the numerator.} \\
&= \lim_{h \to 0} \frac{h(3h+8)}{h} && \text{Factor the numerator.} \\
&= \lim_{h \to 0} (3h + 8) && \text{Cancel the common factor.} \\
&= 8 && \text{Evaluate the limit.}
\end{aligned}$$

The results are the same whether we use **Equation 3.5** or **Equation 3.6**.

3.3 For $f(x) = x^2 + 3x + 2,$ find $f'(1)$.

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}. \tag{3.7}$$

As the values of t approach $a,$ the values of v_{ave} approach the value we call the instantaneous velocity at a. That is, instantaneous velocity at $a,$ denoted $v(a),$ is given by

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}. \tag{3.8}$$

To better understand the relationship between average velocity and instantaneous velocity, see **Figure 3.8**. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.

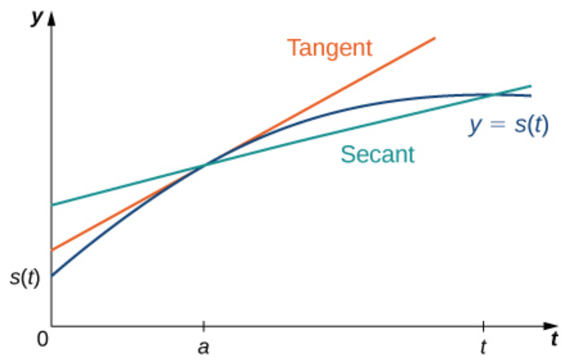


Figure 3.8 The slope of the secant line is the average velocity over the interval $[a,\ t]$. The slope of the tangent line is the instantaneous velocity.

We can use **Equation 3.5** to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using **Equation 3.7**.

Example 3.7

Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ (**Figure 3.9**). Use a table of values to estimate $v(0)$. Check the estimate by using **Equation 3.5**.

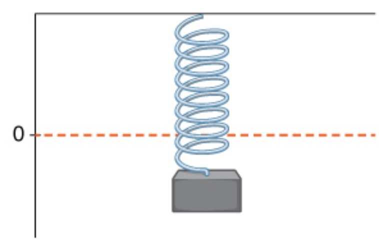


Figure 3.9 A lead weight suspended from a spring in vertical oscillatory motion.

Solution

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching $0,$ as shown in **Table 3.2**.

t	$\frac{\sin t - \sin 0}{t - 0} = \frac{\sin t}{t}$
-0.1	0.998334166
-0.01	0.9999833333
-0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

Table 3.2
Average velocities using values of t approaching 0

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is $0.998334166,$ the average velocity over the time interval $[-0.01, 0]$ is $0.9999833333,$ and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1.$

By using **Equation 3.5**, we can see that

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1.$

3.4 A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64,\ 0 \le t \le 2.$ Find its instantaneous velocity 1 second after it is dropped, using **Equation 3.5**.

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Definition

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example 3.8

Chapter Opener: Estimating Rate of Change of Velocity

Figure 3.10 (credit: modification of work by Codex41, Flickr)

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its acceleration) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

Solution

First observe that 60 mph = 88 ft/s, 100 mph ≈ 146.67 ft/s, 200 mph ≈ 293.33 ft/s, and 229.9 mph ≈ 337.19 ft/s. We can summarize the information in a table.

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

Table 3.3 $v(t)$ at different values of t

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches $19.96,$ as shown in the following table.

t	$\frac{v(t) - v(19.96)}{t - 19.96} = \frac{v(t) - 337.19}{t - 19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

Table 3.4
Average acceleration

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph $(337.19$ ft/s).

Example 3.9

Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70°\text{F}$ at 9 p.m., reaches a low of $60°$ during the night, and rises back to $70°$ by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10,$ where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3).$ Refer to **Equation 3.5**.

$$T'(3) = \lim_{t \to 3} \frac{T(t) - T(3)}{t - 3} \qquad \text{Apply the definition}$$

$$= \lim_{t \to 3} \frac{0.4t^2 - 4t + 70 - 61.6}{t - 3} \qquad \text{Substitute } T(t) = 0.4t^2 - 4t + 70 \text{ and } T(3) = 61.6.$$

$$= \lim_{t \to 3} \frac{0.4t^2 - 4t + 8.4}{t - 3} \qquad \text{Simplify.}$$

$$= \lim_{t \to 3} \frac{0.4(t - 3)(t - 7)}{t - 3} \qquad = \lim_{t \to 3} \frac{0.4(t - 3)(t - 7)}{t - 3}$$

$$= \lim_{t \to 3} 0.4(t - 7) \qquad \text{Cancel.}$$

$$= -1.6 \qquad \text{Evaluate the limit.}$$

The instantaneous rate of change of the temperature at midnight is $-1.6°\text{F}$ per hour.

Example 3.10

Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10{,}000$ dollars. Find the rate of change of profit when $10{,}000$ games are produced. Should the toy company increase or decrease production?

Solution

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x),$ where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

$$P(x) = -0.01x^2 + 300x - 10{,}000.$$

Therefore, evaluating the rate of change of profit gives

$$\begin{aligned} P'(10000) &= \lim_{x \to 10000} \frac{P(x) - P(10000)}{x - 10000} \\ &= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 10000 - 1990000}{x - 10000} \\ &= \lim_{x \to 10000} \frac{-0.01x^2 + 300x - 2000000}{x - 10000} \\ &= 100. \end{aligned}$$

Since the rate of change of profit $P'(10{,}000) > 0$ and $P(10{,}000) > 0,$ the company should increase production.

3.5 A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10.$ The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25),$ the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Section 2.8: The Derivative as a Function

The following videos provide useful examples of the material about the be covered:
blackpenredpen - Why isn't abs(x) differentiable at x=0? (definition of derivative)
The Organic Chemistry Tutor - Higher Order Derivatives

3.2 | The Derivative as a Function

Learning Objectives

3.2.1 Define the derivative function of a given function.
3.2.2 Graph a derivative function from the graph of a given function.
3.2.3 State the connection between derivatives and continuity.
3.2.4 Describe three conditions for when a function does not have a derivative.
3.2.5 Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Definition

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.9}$$

A function $f(x)$ is said to be **differentiable at** $\boldsymbol{a}$ if $f'(a)$ exists. More generally, a function is said to be **differentiable on** $\boldsymbol{S}$ if it is differentiable at every point in an open set S, and a **differentiable function** is one in which $f'(x)$ exists on its domain.

In the next few examples we use **Equation 3.9** to find the derivative of a function.

Example 3.11

Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Start directly with the definition of the derivative function. Use **Equation 3.1**.

$f'(x) = \lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$	Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
$= \lim\limits_{h \to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$	Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.
$= \lim\limits_{h \to 0} \dfrac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$	Multiply the numerators and simplify.
$= \lim\limits_{h \to 0} \dfrac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$	Cancel the h.
$= \dfrac{1}{2\sqrt{x}}$	Evaluate the limit.

Example 3.12

Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution

Follow the same procedure here, but without having to multiply by the conjugate.

$f'(x) = \lim\limits_{h \to 0} \dfrac{\left((x+h)^2 - 2(x+h)\right) - (x^2 - 2x)}{h}$	Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
$= \lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$	Expand $(x+h)^2 - 2(x+h)$.
$= \lim\limits_{h \to 0} \dfrac{2xh - 2h + h^2}{h}$	Simplify.
$= \lim\limits_{h \to 0} \dfrac{h(2x - 2 + h)}{h}$	Factor out h from the numerator.
$= \lim\limits_{h \to 0} (2x - 2 + h)$	Cancel the common factor of h.
$= 2x - 2$	Evaluate the limit.

3.6 Find the derivative of $f(x) = x^2$.

We use a variety of different notations to express the derivative of a function. In **Example 3.12** we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\dfrac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\dfrac{d}{dx}\left(x^2 - 2x\right) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

$$f'(x),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}(f(x)).$$

In place of $f'(a)$ we may also use $\left.\frac{dy}{dx}\right|_{x=a}$ Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx (**Figure 3.11**). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

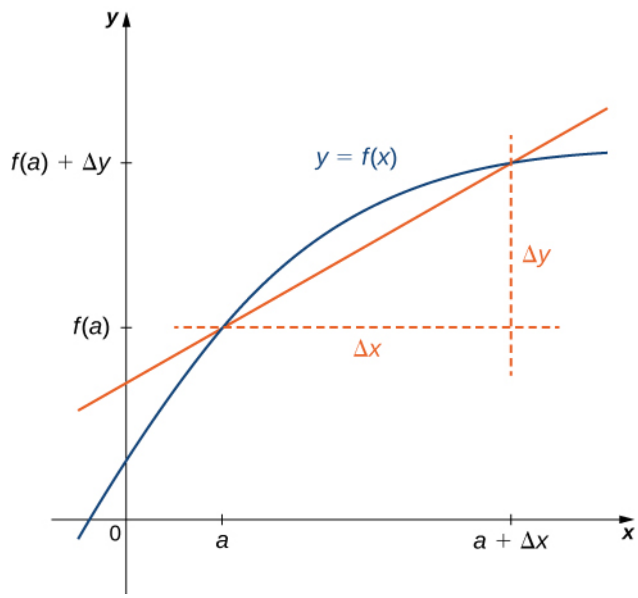


Figure 3.11 The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In **Example 3.11** we found that for $f(x) = \sqrt{x}$, $f'(x) = 1/2\sqrt{x}$. If we graph these functions on the same axes, as in **Figure 3.12**, we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.

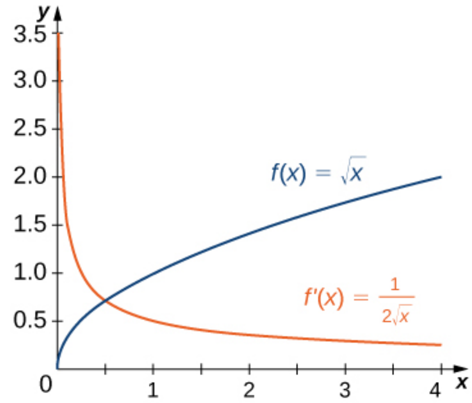


Figure 3.12 The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In **Example 3.12** we found that for $f(x) = x^2 - 2x,\ f'(x) = 2x - 2.$ The graphs of these functions are shown in **Figure 3.13**. Observe that $f(x)$ is decreasing for $x < 1.$ For these same values of $x,\ f'(x) < 0.$ For values of $x > 1,\ f(x)$ is increasing and $f'(x) > 0.$ Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0.$

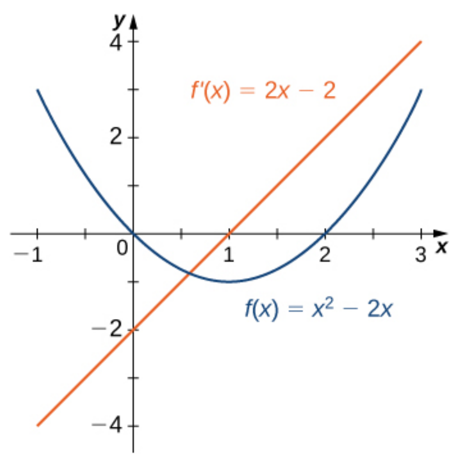
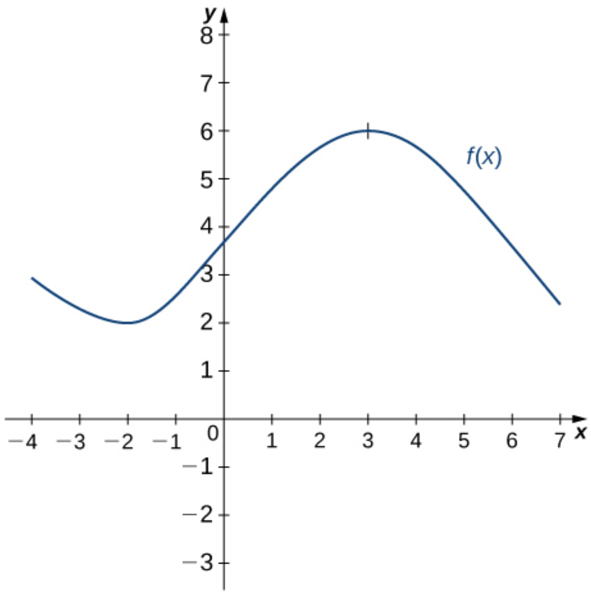


Figure 3.13 The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Example 3.13

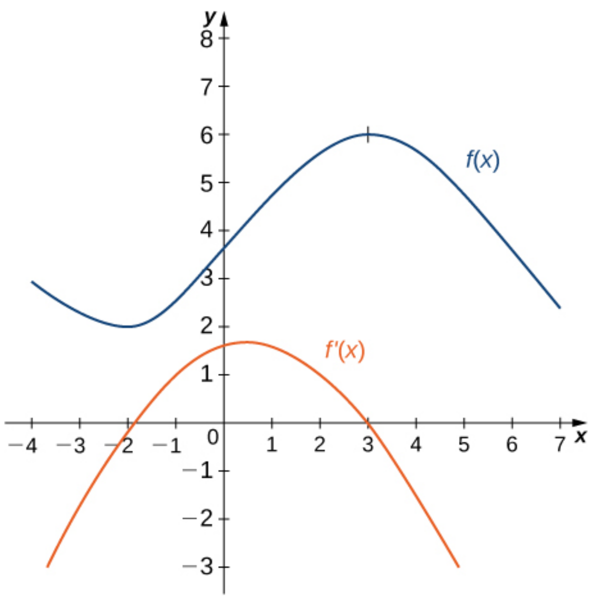
Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.



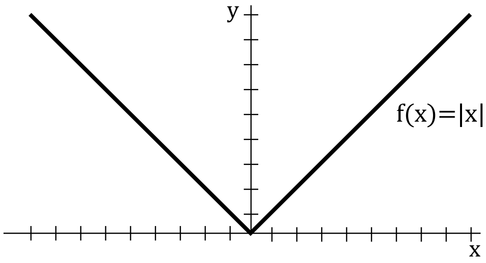
3.7 Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Example 4.10: Derivative of the Absolute Value

Discuss the derivative of the absolute value function $y = f(x) = |x|$.

Solution. If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin.

Source: Lyryx Learning, Calculus: Early Transcendentals, 2016 pg 125



We can summarize this as

$$y' = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ \text{undefined}, & \text{if } x = 0. \end{cases}$$

In particular, the absolute value function $f(x) = |x|$ is *not* differentiable at $x = 0$. ♣

We note that the following theorem can be proved using limits.

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there;

however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Theorem 3.1: Differentiability Implies Continuity

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at $a,$ then f is continuous at $a.$

Proof

If $f(x)$ is differentiable at $a,$ then $f'(a)$ exists and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \to a} f(x) = f(a).$ Thus,

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \big(f(x) - f(a) + f(a)\big) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a)\right) && \text{Multiply and divide } f(x) - f(a) \text{ by } x - a. \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \left(\lim_{x \to a} (x - a)\right) + \lim_{x \to a} f(a) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a).
\end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a),$ we conclude that f is continuous at $a.$

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|.$ This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|,$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

This limit does not exist because

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \text{ and } \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

See **Figure 3.14**.

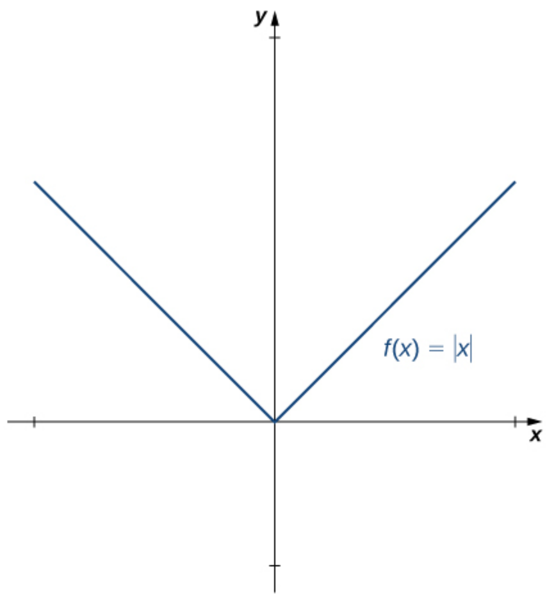


Figure 3.14 The function $f(x) = |x|$ is continuous at 0 but is not differentiable at $0.$

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

$$f'(0) = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 (**Figure 3.15**).

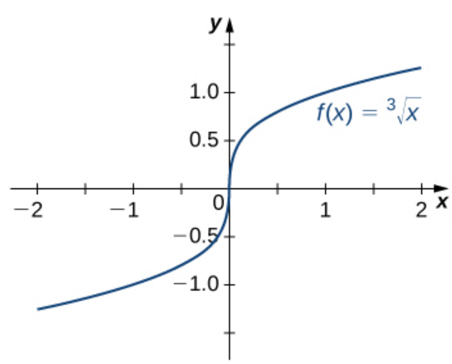


Figure 3.15 The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0.$ It is continuous at 0 but is not differentiable at $0.$

The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at $0.$ We see that

$$f'(0) = \lim_{x \to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero (**Figure 3.16**).

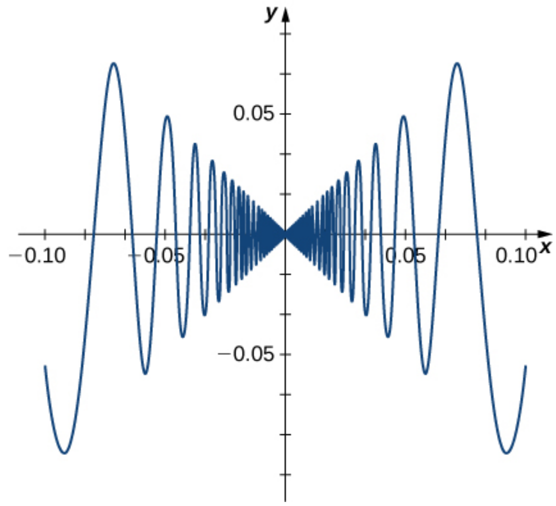


Figure 3.16 The function $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be "smooth" at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.
4. As we saw with $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example 3.14

A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (**Figure 3.17**). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c \text{ if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2} \text{ if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

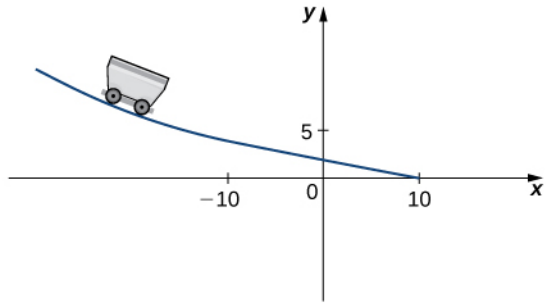


Figure 3.17 For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim_{x \to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x \to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x \to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}
\lim_{x \to -10^-} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x + 10} \\
&= \lim_{x \to -10^-} \frac{\frac{1}{10}x^2 + bx + (10b - 5) - 5}{x + 10} && \text{Substitute } c = 10b - 5. \\
&= \lim_{x \to -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x + 10)} \\
&= \lim_{x \to -10^-} \frac{(x + 10)(x - 10 + 10b)}{10(x + 10)} && \text{Factor by grouping.} \\
&= b - 2.
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{x \to -10^+} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \to -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x + 10} \\
&= \lim_{x \to -10^+} \frac{-(x + 10)}{4(x + 10)} \\
&= -\frac{1}{4}.
\end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10\left(\frac{7}{4}\right) - 5 = \frac{25}{2}$.

3.8 Find values of a and b that make $f(x) = \begin{cases} ax + b & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x),\ f^{(4)}(x),\ldots,\ f^{(n)}(x)$$

$$y''(x),\ y'''(x),\ y^{(4)}(x),\ldots,\ y^{(n)}(x)$$

$$\frac{d^2y}{dx^2},\ \frac{d^3y}{dy^3},\ \frac{d^4y}{dy^4},\ldots,\ \frac{d^ny}{dy^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}.$

Example 3.15

Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution

First find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{\left(2(x+h)^2 - 3(x+h) + 1\right) - (2x^2 - 3x + 1)}{h}$$

Substitute $f(x) = 2x^2 - 3x + 1$ and $f(x+h) = 2(x+h)^2 - 3(x+h) + 1$ into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$= \lim_{h \to 0} \frac{4xh + h^2 - 3h}{h}$$

Simplify the numerator.

$$= \lim_{h \to 0} (4x + h - 3)$$

Factor out the h in the numerator and cancel with the h in the denominator.

$$= 4x - 3$$

Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ with $f'(x)$ in place of $f(x)$.

$$= \lim_{h \to 0} \frac{(4(x+h) - 3) - (4x - 3)}{h}$$

Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.

$$= \lim_{h \to 0} 4$$

Simplify.

$$= 4$$

Take the limit.

3.9 Find $f''(x)$ for $f(x) = x^2$.

Example 3.16

Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t.

Solution

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

$$\begin{aligned} s'(t) &= \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} \\ &= \lim_{h \to 0} \frac{3(t+h)^2 - 4(t+h) + 1 - \left(3t^2 - 4t + 1\right)}{h} \\ &= 6t - 4. \end{aligned}$$

Next,

$$\begin{aligned} s''(t) &= \lim_{h \to 0} \frac{s'(t+h) - s'(t)}{h} \\ &= \lim_{h \to 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

3.10 For $s(t) = t^3$, find $a(t)$.

Chapter 3: Differentiation Rules

Section 3.1: Derivatives of Polynomials and Exponential Functions

The following video provides a useful intuitive explanation of the topics about to be covered:
3Blue1Brown - What's so special about Euler's number e?

3.3 | Differentiation Rules

Learning Objectives

- **3.3.1** State the constant, constant multiple, and power rules.
- **3.3.2** Apply the sum and difference rules to combine derivatives.
- **3.3.3** Use the product rule for finding the derivative of a product of functions.
- **3.3.4** Use the quotient rule for finding the derivative of a quotient of functions.
- **3.3.5** Extend the power rule to functions with negative exponents.
- **3.3.6** Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit. The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated. In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x + h) = c$, so we obtain the following result:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} \frac{0}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

Theorem 3.2: The Constant Rule

Let c be a constant.

If $f(x) = c$, then $f'(c) = 0$.

Alternatively, we may express this rule as

$$\frac{d}{dx}(c) = 0.$$

Example 3.17

Applying the Constant Rule

Find the derivative of $f(x) = 8$.

Solution

This is just a one-step application of the rule:

$$f'(8) = 0.$$

3.11 Find the derivative of $g(x) = -3$.

The Power Rule

We have shown that

$$\frac{d}{dx}\left(x^2\right) = 2x \text{ and } \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example 3.18

Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution

$\frac{d}{dx}\left(x^3\right) = \lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h}$	
$= \lim\limits_{h \to 0} \frac{x^3 + 3x^2 h + 3xh^2 + h^3 - x^3}{h}$	Notice that the fir t term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2 h$. All other terms contain powers of h that are two or greater.
$= \lim\limits_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$= \lim\limits_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$	Factor out the common factor of h.
$= \lim\limits_{h \to 0} (3x^2 + 3xh + h^2)$	After cancelling the common factor of h, the only term not containing h is $3x^2$.
$= 3x^2$	Let h go to 0.

3.12 Find $\frac{d}{dx}\left(x^4\right)$.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

Theorem 3.3: The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

$$\text{Since } (x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h \to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \ldots + nxh^{n-1} + h^n\right) \\ &= nx^{n-1}. \end{aligned}$$

□

Example 3.19

Applying the Power Rule

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution

Using the power rule with $n = 10,$ we obtain

$$f'(x) = 10x^{10-1} = 10x^9.$$

3.13 Find the derivative of $f(x) = x^7$.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Theorem 3.4: Sum, Difference, and Constant Multiple Rules

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) + g(x),\ j'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of g:

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

$$\text{for } j(x) = f(x) - g(x),\ j'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant *k* multiplied by a function *f* is the same as the constant multiplied by the derivative:

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x));$$

that is,

$$\text{for } j(x) = kf(x),\ j'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $j(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - j(x)}{h}.$$

By substituting $j(x+h) = f(x+h) + g(x+h)$ and $j(x) = f(x) + g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}.$$

Rearranging and regrouping the terms, we have

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right) + \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right) = f'(x) + g'(x).$$

□

Example 3.20

Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

Solution

We use the power rule directly:

$$g'(x) = \frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of

$f(x)$. This relationship is illustrated in **Figure 3.18**.

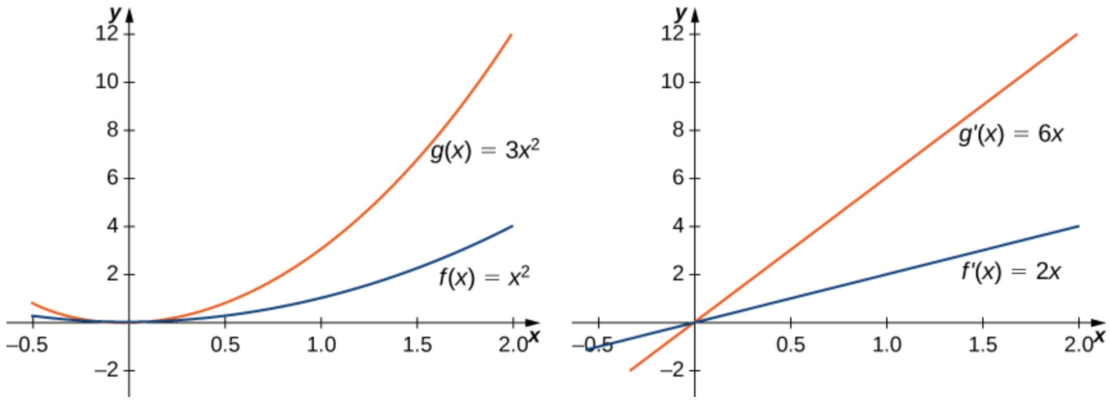


Figure 3.18 The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example 3.21

Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5 + 7\right) & \\ &= \frac{d}{dx}\left(2x^5\right) + \frac{d}{dx}(7) & \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}\left(x^5\right) + \frac{d}{dx}(7) & \text{Apply the constant multiple rule.} \\ &= 2\left(5x^4\right) + 0 & \text{Apply the power rule and the constant rule.} \\ &= 10x^4. & \text{Simplify.} \end{aligned}$$

3.14 Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Example 3.22

Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

$$y = -2x + 5.$$

3.15 Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Corresponds to CH3, Section 3.1, Example 6 in Stewart Calculus

Where do $f(x)x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have horizontal tangent lines?
SOLUTION
$f'(x) = 2x - 10$. $f'(x) = 0$ when $2x - 10 = 0$ so when $\mathbf{x = 5}$.
$g'(x) = 3x^2 - 12$. $g'(x) = 0$ when $3x^2 - 12 = 0$ so $x^2 = 4$ and $x = -2; x = 2$

Source: Hoffman, *Contemporary Calculus*; Section 2.2 - Derivatives: Properties and Formulas

3.9 | Derivatives of Exponential and Logarithmic Functions

Learning Objectives

3.9.1 Find the derivative of exponential functions.
3.9.2 Find the derivative of logarithmic functions.
3.9.3 Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in **Introduction to Functions and Graphs**, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x,\ b > 0,$ is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of $b^n,$ where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1,\ b^{-n} = \frac{1}{b^n},$ for a positive integer $n,$ and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and $t.$ These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the *continuity* of $B(x) = b^x,\ b > 0,$ we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4,\ 4^{3.1} < 4^\pi < 4^{3.2},\ 4^{3.14} < 4^\pi < 4^{3.15},$$
$$4^{3.141} < 4^\pi < 4^{3.142},\ 4^{3.1415} < 4^\pi < 4^{3.1416},\ldots.$$

As we see in the following table, $4^\pi \approx 77.88.$

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Table 3.7 Approximating a Value of 4^{π}

We also assume that for $B(x) = b^x, b > 0,$ the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1.$ We define e to be this unique value, as we did in **Introduction to Functions and Graphs**. **Figure 3.33** provides graphs of the functions $y = 2^x, y = 3^x, y = 2.7^x,$ and $y = 2.8^x.$ A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.

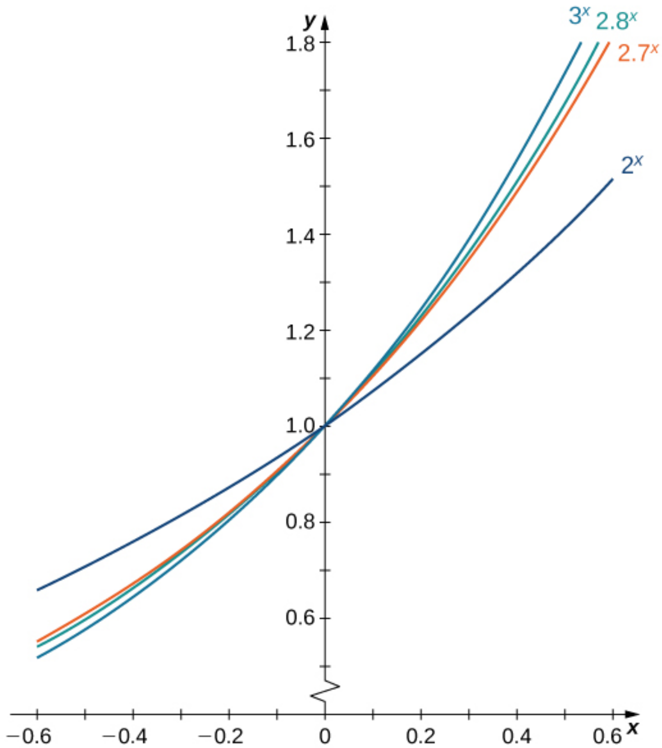


Figure 3.33 The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Table 3.8 Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in **Figure 3.34**. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

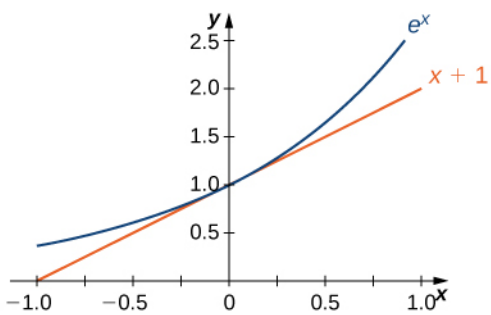


Figure 3.34 The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x, b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}. \quad (3.28)$$

Turning to $B'(x)$, we obtain the following.

$$\begin{aligned}
B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of he derivative.} \\
&= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
&= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} && \text{Factor out } b^x. \\
&= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at $0,$ $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0). \tag{3.29}$$

For $E(x) = e^x,$ $E'(0) = 1.$ Thus, we have $E'(x) = e^x.$ (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x,$ $b > 0,$ will be derived later.)

Section 3.2: The Product and Quotient Rules

The Product Rule

Source: OpenStax, Calculus Volume 1, 2019

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2,$ whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$

Theorem 3.5: Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

$$\text{if } j(x) = f(x)g(x), \text{ then } j'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \to 0} g(x+h) = g(x).$

By applying the limit definition of the derivative to $j(x) = f(x)g(x)$, we obtain

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$j'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \lim_{h \to 0}\left(\frac{f(x)g(x+h) - f(x)g(x)}{h}\right).$$

Rearranging, we obtain

$$j'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h)\right) + \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h} \cdot f(x)\right).$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

$$j'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example 3.23

Applying the Product Rule to Functions at a Point

For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution

Since $j(x) = f(x)g(x)$, $j'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

Example 3.24

Applying the Product Rule to Binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

$$j'(x) = f'(x)g(x) + g'(x)f(x) = (2x)\left(3x^3 - 5x\right) + (9x^2 - 5)(x^2 + 2).$$

Simplifying, we have

$$j'(x) = 15x^4 + 3x^2 - 10.$$

Source: OpenStax, Calculus Volume 1, 2019 pg 255

To check, we see that $j(x) = 3x^5 + x^3 - 10x$ and, consequently, $j'(x) = 15x^4 + 3x^2 - 10.$

3.16 Use the product rule to obtain the derivative of $j(x) = 2x^5\left(4x^2 + x\right).$

Example 4.24: Derivative of a Product of Functions

Find the derivative of $h(x) = (3x-1)(2x+3)$.

Solution. One way to do this question is to expand the expression. Alternatively, we use the product rule with $f(x) = 3x-1$ and $g(x) = 2x+3$. Note that $f'(x) = 3$ and $g'(x) = 2$, so,

$$h'(x) = (3)\cdot(2x+3)+(3x-1)\cdot(2) = 6x+9+6x-2 = 12x+7.$$

♣

Example 4.26: Derivative of a Quotient of Functions

Find the derivative of $h(x) = \dfrac{3x-1}{2x+3}$.

Solution. By the quotient rule (using $f(x) = 3x-1$ and $g(x) = 2x+3$) we have:

$$h'(x) = \frac{\frac{d}{dx}(3x-1)\cdot(2x+3) - (3x-1)\cdot\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

$$= \frac{3(2x+3)-(3x-1)(2)}{(2x+3)^2} = \frac{11}{(2x+3)^2}.$$

Source: Lyryx Learning, Calculus; Early Transcendentals, 2017 pg 133

♣

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

$$\frac{d}{dx}\left(x^2\right) = 2x, \text{ not } \frac{\frac{d}{dx}\left(x^3\right)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

Theorem 3.6: The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

That is,

$$\text{if } j(x) = \frac{f(x)}{g(x)}, \text{ then } j'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

The proof of the **quotient rule** is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example 3.25

Applying the Quotient Rule

Use the quotient rule to find the derivative of $k(x) = \frac{5x^2}{4x + 3}$.

Solution

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$. Substituting into the quotient rule, we have

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x + 3) - 4(5x^2)}{(4x + 3)^2}.$$

Simplifying, we obtain

$$k'(x) = \frac{20x^2 + 30x}{(4x+3)^2}.$$

3.17 Find the derivative of $h(x) = \dfrac{3x+1}{4x-3}.$

APPENDIX B | TABLE OF DERIVATIVES

General Formulas

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

4. $\frac{d}{dx}(x^n) = nx^{n-1}, \text{ for real numbers } n$

5. $\frac{d}{dx}(cf(x)) = cf'(x)$

6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Section 3.3: Derivatives of Trigonometric Functions

The following videos provide useful explanation of an important trigonometric limit that is not in the open source textbooks used:
Khan Academy - Proof: lim (sin x)/x || Limits || Differential Calculus

3.5 | Derivatives of Trigonometric Functions

Learning Objectives

3.5.1 Find the derivatives of the sine and cosine function.
3.5.2 Find the derivatives of the standard trigonometric functions.
3.5.3 Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. We see that by using $h = 0.01$,

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting $D(x) = \frac{\sin(x + 0.01) - \sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ (**Figure 3.25**).

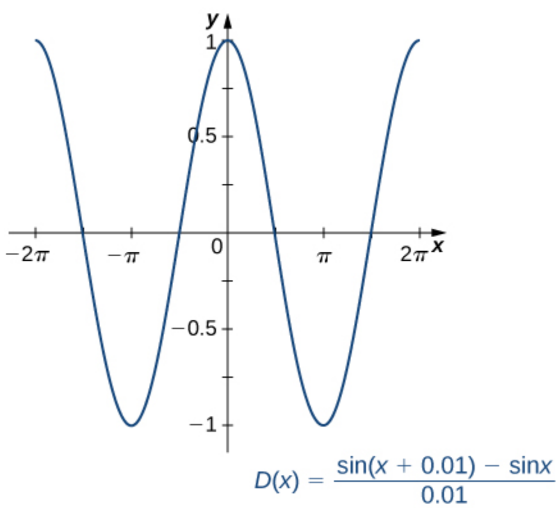


Figure 3.25 The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Theorem 3.8: The Derivatives of sin *x* and cos *x*

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

$$\frac{d}{dx}(\sin x) = \cos x \qquad (3.11)$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad (3.12)$$

Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in **Introduction to Limits**:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in **Figure 3.26**.

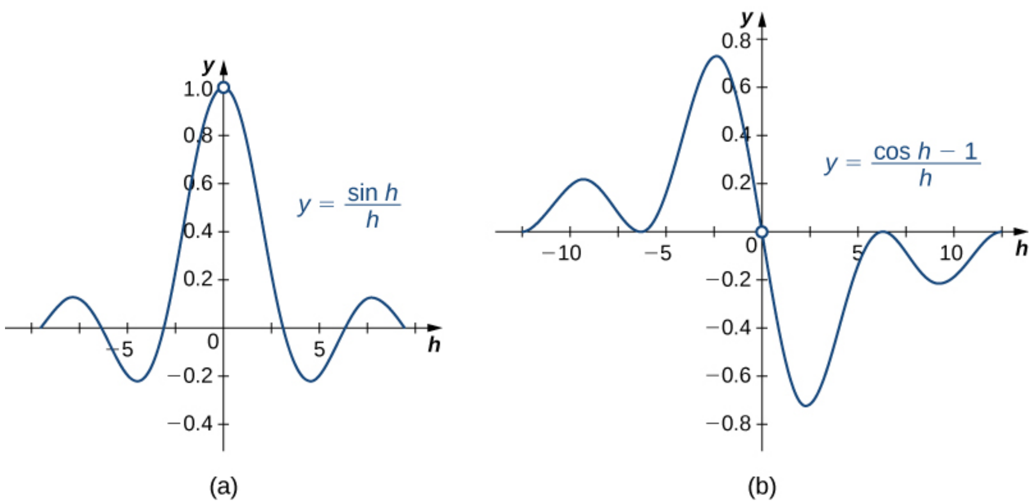


Figure 3.26 These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Apply the definition of he derivative.} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{Use trig identity for the sine of the sum of two angles.} \\
&= \lim_{h \to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) && \text{Regroup.} \\
&= \lim_{h \to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) && \text{Factor out } \sin x \text{ and } \cos x. \\
&= \sin x(0) + \cos x(1) && \text{Apply trig limit formulas.} \\
&= \cos x && \text{Simplify.}
\end{aligned}$$

□

Figure 3.27 shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where $f(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.

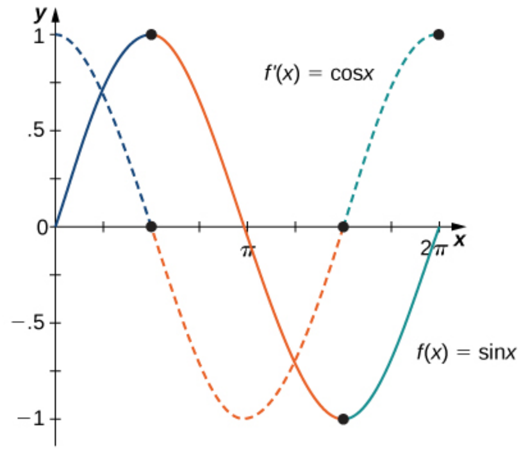


Figure 3.27 Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$ that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs.

Example 3.39

Differentiating a Function Containing sin *x*

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution

Using the product rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(5x^3\right) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

3.25 Find the derivative of $f(x) = \sin x \cos x$.

Example 4.29: Derivative of Product of Trigonometric Functions

Find the derivative of $f(x) = \sin x \tan x$.

Solution. Using the Product Rule we obtain

$$f'(x) = \cos x \tan x + \sin x \sec^2 x.$$

♣

Example 3.40

Finding the Derivative of a Function Containing cos *x*

Source: OpenStax, Calculus Volume 1, 2019 pg 279

Find the derivative of $g(x) = \dfrac{\cos x}{4x^2}$.

Solution

By applying the quotient rule, we have

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{\left(4x^2\right)^2}.$$

Simplifying, we obtain

$$\begin{aligned} g'(x) &= \frac{-4x^2 \sin x - 8x \cos x}{16x^4} \\ &= \frac{-x \sin x - 2\cos x}{4x^3}. \end{aligned}$$

3.26 Find the derivative of $f(x) = \dfrac{x}{\cos x}$.

Example 3.41

An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

$$s'(t) = 2\cos t - 1,$$

so we must solve

$$2\cos t - 1 = 0 \text{ for } 0 \le t \le 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

3.27 A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example 3.42

The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1,$ by the Pythagorean theorem, we now have

$$f'(x) = \frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

$$f'(x) = \sec^2 x.$$

3.28 Find the derivative of $f(x) = \cot x$.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Theorem 3.9: Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

$$\frac{d}{dx}(\tan x) = \sec^2 x \tag{3.13}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \tag{3.14}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \tag{3.15}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x. \tag{3.16}$$

Example 3.43

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

$$f\left(\frac{\pi}{4}\right) = \cot\frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example 3.44

Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x\tan x$.

Solution

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x\tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x\cot x$, and by applying the product rule to the second term we obtain

$$\frac{d}{dx}(x\tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

$$f'(x) = -\csc x\cot x + \tan x + x\sec^2 x.$$

3.29 Find the derivative of $f(x) = 2\tan x - 3\cot x$.

3.30 Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example 3.45

Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution

Each step in the chain is straightforward:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2 y}{dx^2} &= -\sin x \\ \frac{d^3 y}{dx^3} &= -\cos x \\ \frac{d^4 y}{dx^4} &= \sin x. \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of sin x equals sin x, so

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$
$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

3.31 For $y = \cos x$, find $\frac{d^4 y}{dx^4}$.

Example 3.46

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

3.32 For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Example 3.47

An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution

First find $v(t) = s'(t)$:

$$v(t) = s'(t) = -\cos t.$$

Thus,

$$v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

$$a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is travelling. Consequently, the particle is slowing down.

3.33 A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Section 3.4: The Chain Rule

The following videos provide useful intuitive explanations of the topics about to be covered:
3Blue1Brown - Visualizing the chain rule and the product rule
Khan Academy - Chain Rule

3.6 | The Chain Rule

Learning Objectives

- **3.6.1** State the chain rule for the composition of two functions.
- **3.6.2** Apply the chain rule together with the power rule.
- **3.6.3** Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- **3.6.4** Recognize the chain rule for a composition of three or more functions.
- **3.6.5** Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions $(x^n, \sin x, \cos x, \text{etc.})$ as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}.$ In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the **chain rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3).$ We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x. Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3).$ This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3).$ First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u = x^3,$ is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3).$

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x - a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

$$h'(a) = \lim_{x \to a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a.$ That is,

$$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3) = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \to a,\ u \to a^3$:

$$\begin{aligned}\lim_{x \to a} \frac{\sin\left(x^3\right) - \sin\left(a^3\right)}{x^3 - a^3} &= \lim_{u \to a^3} \frac{\sin u - \sin\left(a^3\right)}{u - a^3} \\ &= \frac{d}{du}(\sin u)_{u = a^3} \\ &= \cos\left(a^3\right).\end{aligned}$$

Thus, $h'(a) = \cos\left(a^3\right) \cdot 3a^2.$

In other words, if $h(x) = \sin\left(x^3\right),$ then $h'(x) = \cos\left(x^3\right) \cdot 3x^2.$ Thus, if we think of $h(x) = \sin\left(x^3\right)$ as the composition $(f \circ g)(x) = f(g(x))$ where $f(x) =$ sin x and $g(x) = x^3,$ then the derivative of $h(x) = \sin\left(x^3\right)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3.$ At this point, we anticipate that for $h(x) = \sin(g(x)),$ it is quite likely that $h'(x) = \cos(g(x))g'(x).$ As we determined above, this is the case for $h(x) = \sin\left(x^3\right).$

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x),$ the derivative of the composite function

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

$$h'(x) = f'(g(x))g'(x). \tag{3.17}$$

Alternatively, if y is a function of $u,$ and u is a function of $x,$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Watch an **animation (http://www.openstaxcollege.org/l/20_chainrule2)** of the chain rule.

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f(g(x)),$ begin by identifying $f(x)$ and $g(x).$
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x)).$
3. Find $g'(x).$
4. Write $h'(x) = f'(g(x)) \cdot g'(x).$

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = (g(x))^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = (g(x))^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'(g(x)) = n(g(x))^{n-1}$. This leads us to the derivative of a power function using the chain rule,

$$h'(x) = n(g(x))^{n-1} g'(x)$$

Rule: Power Rule for Composition of Functions

For all values of x for which the derivative is defined, if

$$h(x) = (g(x))^n.$$

Then

$$h'(x) = n(g(x))^{n-1} g'(x). \tag{3.18}$$

Example 4.30: Chain Rule

Compute the derivative of $\sqrt{625 - x^2}$.

Source: Lyryx Learning, Calculus: Early Transcendentals, 2017, pg 137

Solution. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x)=\sqrt{x}$, $g(x)=625-x^2$. We know that $f'(x)=(1/2)x^{-1/2}$, so $f'(g(x))=(1/2)(625-x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x)=-2x$ we have

$$f'(g(x))g'(x)=\frac{1}{2\sqrt{625-x^2}}(-2x)=\frac{-x}{\sqrt{625-x^2}}.$$

♣

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

$$k(x) = h(f(g(x))).$$

Then, applying the chain rule once we obtain

$$k'(x) = \frac{d}{dx}(h(f(g(x))) = h'(f(g(x))) \cdot \frac{d}{dx} f((g(x))).$$

Applying the chain rule again, we obtain

$$k'(x) = h'(f(g(x))f'(g(x))g'(x)).$$

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

$$k(x) = h(f(g(x))),$$

then

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example 3.55

Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution

First, rewrite $k(x)$ as

$$k(x) = \left(\cos\left(7x^2 + 1\right)\right)^4.$$

Then apply the chain rule several times.

$$\begin{aligned} k'(x) &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(\frac{d}{dx}\left(\cos\left(7x^2+1\right)\right)\right) && \text{Apply the chain rule.} \\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)\left(\frac{d}{dx}\left(7x^2+1\right)\right) && \text{Apply the chain rule.} \\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)(14x) && \text{Apply the chain rule.} \\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify.} \end{aligned}$$

3.38 Find the derivative of $h(x) = \sin^6(x^3)$.

Source: OpenStax, Calculus Volume 1, 2019, pg 293

Example 4.32: Derivative of Quotient

Compute the derivative of

$$f(x) = \frac{x^2-1}{x\sqrt{x^2+1}}.$$

Solution. The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$\begin{aligned} f'(x) &= \frac{(x^2-1)'x\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)} \\ &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \end{aligned}$$

Now we need to compute the derivative of $x\sqrt{x^2+1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2+1} = x\frac{d}{dx}\sqrt{x^2+1}+\sqrt{x^2+1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

Source: Lyryx Learning, Calculus: Early Transcendentals, 2017, pg 138

And putting it all together:

$$\begin{aligned} f'(x) &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \\ &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)\left(x\dfrac{x}{\sqrt{x^2+1}}+\sqrt{x^2+1}\right)}{x^2(x^2+1)}. \end{aligned}$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left. ♣

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = g(u)$. Thus,

$$h'(x) = \frac{dy}{dx},\ f'(g(x)) = f'(u) = \frac{dy}{du} \text{ and } g'(x) = \frac{du}{dx}.$$

Consequently,

$$\frac{dy}{dx} = h'(x) = f'(g(x))g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 3.58

Taking a Derivative Using Leibniz's Notation, Example 1

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

Source: OpenStax, Calculus Volume 1, 2019, pg 295

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\
&= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\
&= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\
&= \frac{10x^4}{(3x+2)^6} && \text{Simplify.}
\end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example 3.51

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution

Think of $h(x) = \cos(g(x))$ as $f(g(x))$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$. we have $f'(g(x)) = -\sin(g(x))$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) && \text{Apply the chain rule.} \\ &= -\sin(g(x))g'(x) && \text{Substitute } f'(g(x)) = -\sin(g(x)). \end{aligned}$$

Thus, the derivative of $h(x) = \cos(g(x))$ is given by $h'(x) = -\sin(g(x))g'(x)$.

In the following example we apply the rule that we have just derived.

Example 3.52

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos\left(5x^2\right)$.

Solution

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

$$\begin{aligned} h'(x) &= -\sin\left(5x^2\right) \cdot 10x \\ &= -10x\sin\left(5x^2\right). \end{aligned}$$

Example 3.53

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec\left(4x^5 + 2x\right)$.

Solution

Apply the chain rule to $h(x) = \sec(g(x))$ to obtain

$$h'(x) = \sec(g(x)\tan(g(x))g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

$$\begin{aligned} h'(x) &= \sec\left(4x^5 + 2x\right)\tan\left(4x^5 + 2x\right)\left(20x^4 + 2\right) \\ &= (20x^4 + 2)\sec\left(4x^5 + 2x\right)\tan\left(4x^5 + 2x\right). \end{aligned}$$

3.36 Find the derivative of $h(x) = \sin(7x + 2)$.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in **Example 3.51** and **Example 3.53**. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Theorem 3.10: Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

$$\begin{aligned}
\frac{d}{dx}(\sin(g(x)) &= \cos(g(x))g'(x) & \frac{d}{dx}\sin u &= \cos u\frac{du}{dx} \\
\frac{d}{dx}(\cos(g(x)) &= -\sin(g(x))g'(x) & \frac{d}{dx}\cos u &= -\sin u\frac{du}{dx} \\
\frac{d}{dx}(\tan(g(x)) &= \sec^2(g(x))g'(x) & \frac{d}{dx}\tan u &= \sec^2 u\frac{du}{dx} \\
\frac{d}{dx}(\cot(g(x)) &= -\csc^2(g(x))g'(x) & \frac{d}{dx}\cot u &= -\csc^2 u\frac{du}{dx} \\
\frac{d}{dx}(\sec(g(x)) &= \sec(g(x)\tan(g(x))g'(x) & \frac{d}{dx}\sec u &= \sec u\tan u\frac{du}{dx} \\
\frac{d}{dx}(\csc(g(x)) &= -\csc(g(x))\cot(g(x))g'(x) & \frac{d}{dx}\csc u &= -\csc u\cot u\frac{du}{dx}.
\end{aligned}$$

Example 3.54

Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution

First apply the product rule, then apply the chain rule to each term of the product.

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left((2x+1)^5\right)\cdot(3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right)\cdot(2x+1)^5 && \text{Apply the product rule.} \\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{Apply the chain rule.} \\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\
&= (2x+1)^4(3x-2)^6(10(3x-7)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6. \\
&= (2x+1)^4(3x-2)^6(72x-49) && \text{Simplify.}
\end{aligned}$$

3.37 Find the derivative of $h(x) = \dfrac{x}{(2x+3)^3}$.

Example 4.33: Chain of Composition

Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$.

Solution. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x) = 1+\sqrt{1+\sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}(1+\sqrt{x})^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{1}{2}(1+\sqrt{x})^{-1/2}\frac{1}{2}x^{-1/2}. \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}\end{aligned}$$

♣

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

Example 6: Determine $\mathbf{D}(e^{\cos(x)})$ using each form of the Chain Rule.

Solution: Using the Leibniz notation: $y = e^u$ and $u = \cos(x)$. $dy/du = e^u$ and $du/dx = -\sin(x)$ so

$$dy/dx = (dy/du)\cdot(du/dx) = (e^u)\cdot(-\sin(x)) = -\sin(x) \cdot e^{\cos(x)} .$$

The function $e^{\cos(x)}$ is also the composition of $f(x) = e^x$ with $g(x) = \cos(x)$, so

$\mathbf{D}(e^{\cos(x)}) = \mathbf{f}'(g(x))\cdot \mathbf{g}'(x)$ by the Chain Rule

$= \mathbf{e}^{g(x)}\cdot(\mathbf{-\sin}(x))$ since $\mathbf{D}(e^x) = e^x$ and $\mathbf{D}(\cos(x)) = -\sin(x)$

$= -\sin(x) \cdot e^{\cos(x)}$.

Section 3.5: Implicit Differentiation

The following video provides a useful intuitive explanation of the topics about to be covered:
3Blue1Brown - Implicit differentiation - what's going on here?

3.8 | Implicit Differentiation

Learning Objectives

- 3.8.1 Find the derivative of a complicated function by using implicit differentiation.
- 3.8.2 Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$y = \sqrt{25 - x^2}$ and $y = \begin{cases} \sqrt{25 - x^2} & \text{if } -25 \le x < 0 \\ -\sqrt{25 - x^2} & \text{if } 0 \le x \le 25 \end{cases}$, which are illustrated in **Figure 3.30**, are just three of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.

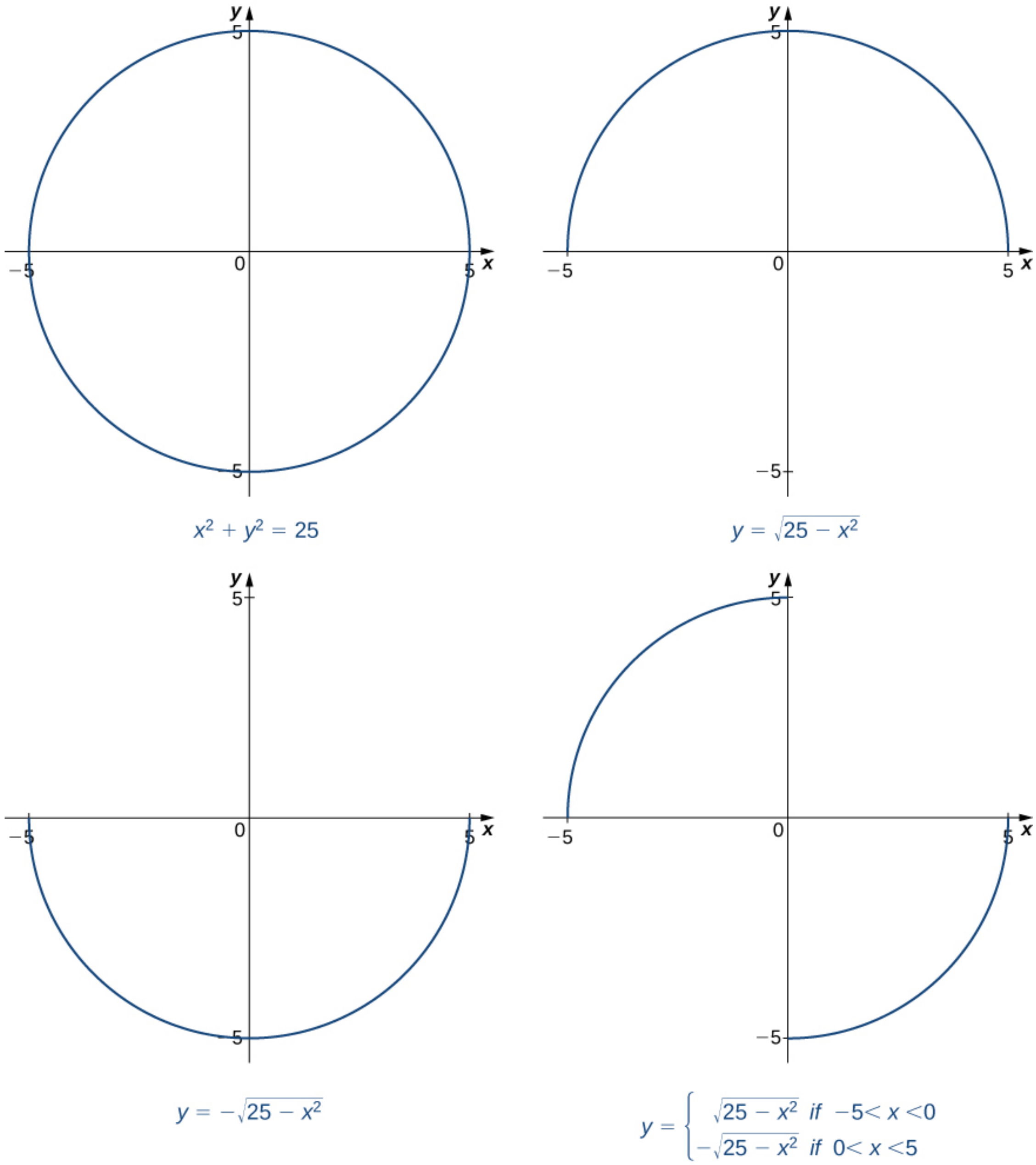


Figure 3.30 The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\frac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example 3.68

Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution

Follow the steps in the problem-solving strategy.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$ Step 1. Diffe entiate both sides of the equation.

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$ Step 1.1. Use the sum rule on the left. On the right $\frac{d}{dx}(25) = 0$.

$$2x + 2y\frac{dy}{dx} = 0$$ Step 1.2. Take the derivatives, so $\frac{d}{dx}\left(x^2\right) = 2x$ and $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x$$ Step 2. Keep the terms with $\frac{dy}{dx}$ on the left. Move the remaining terms to the right.

$$\frac{dy}{dx} = -\frac{x}{y}$$ Step 4. Divide both sides of the equation by $2y$. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example 3.69

Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution

$\frac{d}{dx}\left(x^3 \sin y + y\right)$	$=$	$\frac{d}{dx}(4x + 3)$	Step 1: Diffe entiate both sides of the equation.
$\frac{d}{dx}\left(x^3 \sin y\right) + \frac{d}{dx}(y)$	$=$	4	Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x + 3) = 4$.
$\left(\frac{d}{dx}\left(x^3\right) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3\right) + \frac{dy}{dx}$	$=$	4	Step 1.2: Use the product rule to fin $\frac{d}{dx}\left(x^3 \sin y\right)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.
$3x^2 \sin y + \left(\cos y \frac{dy}{dx}\right) \cdot x^3 + \frac{dy}{dx}$	$=$	4	Step 1.3: We know $\frac{d}{dx}\left(x^3\right) = 3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$.
$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx}$	$=$	$4 - 3x^2 \sin y$	Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.
$\frac{dy}{dx}\left(x^3 \cos y + 1\right)$	$=$	$4 - 3x^2 \sin y$	Step 3: Factor out $\frac{dy}{dx}$ on the left.
$\frac{dy}{dx}$	$=$	$\frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$	Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3 \cos y + 1$.

The information covered in the following video, roughly corresponds to the information covered in Example 3 of this section in the Stewart textbook:
Khan Academy - "Trig Implicit Differentiation Example"

Example 3.70

Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$.

Solution

In **Example 3.68**, we showed that $\frac{dy}{dx} = -\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

Source: OpenStax, Calculus Volume 1, 2019

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dy}\left(-\frac{x}{y}\right) && \text{Diffe entiate both sides of } \frac{dy}{dx} = -\frac{x}{y}. \\
&= -\frac{\left(1 \cdot y - x\frac{dy}{dx}\right)}{y^2} && \text{Use the quotient rule to fin } \frac{d}{dy}\left(-\frac{x}{y}\right). \\
&= \frac{-y + x\frac{dy}{dx}}{y^2} && \text{Simplify.} \\
&= \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} && \text{Substitute } \frac{dy}{dx} = -\frac{x}{y}. \\
&= \frac{-y^2 - x^2}{y^3} && \text{Simplify.}
\end{aligned}$$

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

3.48 Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example 3.71

Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In **Example 3.68**, we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\frac{dy}{dx}\Big|_{(3,\,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ (**Figure 3.31**).

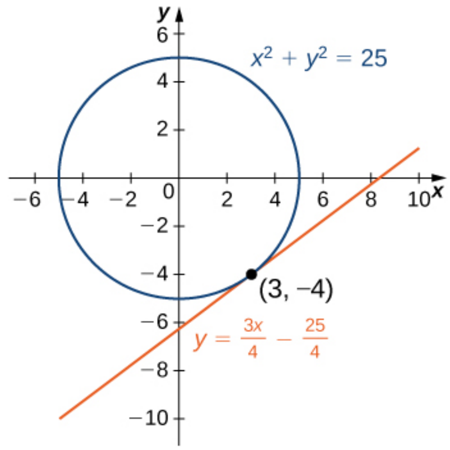


Figure 3.31 The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at the point (3, −4).

Example 3.72

Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (**Figure 3.32**). This curve is known as the folium (or leaf) of Descartes.

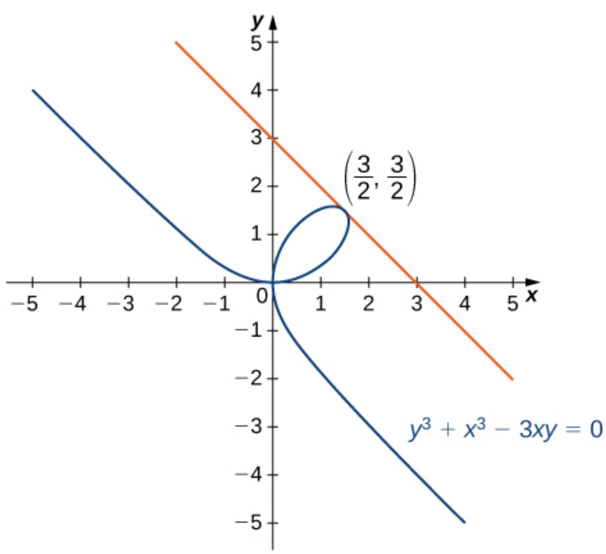


Figure 3.32 Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Solution

Begin by finding $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}\left(y^3 + x^3 - 3xy\right) &= \frac{d}{dx}(0) \\ 3y^2\frac{dy}{dx} + 3x^2 - \left(3y + \frac{dy}{dx}3x\right) &= 0 \\ \frac{dy}{dx} &= \frac{3y - 3x^2}{3y^2 - 3x}. \end{aligned}$$

Next, substitute $\left(\frac{3}{2}, \frac{3}{2}\right)$ into $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ to find the slope of the tangent line:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

$$y = -x + 3.$$

Example 3.73

Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100.$ The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x-axis toward $(0, 0).$ If the rocket fires a missile when it is located at $\left(3, \frac{8}{3}\right),$ where will it intersect the x-axis?

Solution

To solve this problem, we must determine where the line tangent to the graph of $4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{3}\right)$ intersects the x-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

$$8x + 50y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx},$ we have

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3, \frac{8}{3}\right)} = -\frac{9}{50}.$ The equation of the tangent line is $y = -\frac{9}{50}x + \frac{183}{200}.$ To determine where the line intersects the x-axis, solve $0 = -\frac{9}{50}x + \frac{183}{200}$. The solution is $x = \frac{61}{3}$. The missile intersects the x-axis at the point $\left(\frac{61}{3}, 0\right)$.

3.49 Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

The information covered in the following video, roughly corresponds to the information covered in Example 5 of this section in the Stewart textbook:
blackpenredpen - "Implicit Differentiation, $arctan(x^2y) = x + xy^2$"

Chapter 3.6: Derivatives of Logarithmic Functions

This section borrows from multiple sources:

- the first page, taken from OpenStax, provides some important formulas for derivatives of logarithmic functions. This page is followed by a link to a video, providing examples of how to apply these formulas.
- the next few pages, taken from Lyryx Learning, provide an explanation of Logarithmic differentiation
- Another explanation of logarithmic differentiation (along with more examples), taken from OpenStax, is provided.

Exponential and Logarithmic Functions

Source: OpenStax, Calculus Volume 1, 2019, pg 770

21. $\frac{d}{dx}(e^x) = e^x$

22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

23. $\frac{d}{dx}(b^x) = b^x \ln b$

24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

This video provides examples of how to differentiate logarithmic functions:
The Organic Chemistry Tutor - "Derivative of Logarithmic Functions"

Logarithmic Differentiation

Previously we've seen how to do the derivative of a number to a function $(a^{f(x)})'$, and also a function to a number $[(f(x))^n]'$. But what about the derivative of a function to a function $[(f(x))^{g(x)}]'$?

In this case, we use a procedure known as **logarithmic differentiation**.

Steps for Logarithmic Differentiation

- Take ln of both sides of $y = f(x)$ to get $\ln y = \ln f(x)$ and simplify using logarithm properties,
- Differentiate implicitly with respect to x and solve for $\dfrac{dy}{dx}$,
- Replace y with its function of x (i.e., $f(x)$).

Example 4.46: Logarithmic Differentiation

Differentiate $y = x^x$.

Solution. We take ln of both sides:

$$\ln y = \ln x^x.$$

Using log properties we have:

$$\ln y = x \ln x.$$

Differentiating implicitly gives:

$$\frac{y'}{y} = (1)\ln x + x\frac{1}{x}.$$

$$\frac{y'}{y} = \ln x + 1.$$

Solving for y' gives:

$$y' = y(1 + \ln x).$$

Replace $y = x^x$ gives:

$$y' = x^x(1 + \ln x).$$

Another method to find this derivative is as follows:

$$\begin{aligned} \frac{d}{dx}x^x &= \frac{d}{dx}e^{x \ln x} \\ &= \left(\frac{d}{dx}x \ln x\right) e^{x \ln x} \\ &= (x\frac{1}{x} + \ln x)x^x \\ &= (1 + \ln x)x^x \end{aligned}$$

♣

In fact, logarithmic differentiation can be used on more complicated products and quotients (not just when dealing with functions to the power of functions).

Example 4.47: Logarithmic Differentiation

Differentiate (assuming $x > 0$*):*

$$y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}.$$

Solution. Using product & quotient rules for this problem is a complete nightmare! Let's apply logarithmic differentiation instead. Take ln of both sides:

$$\ln y = \ln\left(\frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\right).$$

Applying log properties:

$$\ln y = \ln\left((x+2)^3(2x+1)^9\right) - \ln\left(x^8(3x+1)^4\right).$$

$$\ln y = \ln\left((x+2)^3\right) + \ln\left((2x+1)^9\right) - \left[\ln\left(x^8\right) + \ln\left((3x+1)^4\right)\right].$$

$$\ln y = 3\ln(x+2) + 9\ln(2x+1) - 8\ln x - 4\ln(3x+1).$$

Now, differentiating implicitly with respect to x gives:

$$\frac{y'}{y} = \frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}.$$

Solving for y' gives:

$$y' = y\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

Replace $y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}$ gives:

$$y' = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^{\pi}$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 3.81

Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$$\begin{aligned}
\ln y &= \ln\left(2x^4+1\right)^{\tan x} && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= \tan x \ln\left(2x^4+1\right) && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x && \text{Step 3. Diffe entiate both sides. Use the product rule on the right.}\\
\frac{dy}{dx} &= y\cdot\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right) && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot\tan x\right) && \text{Step 5. Substitute } y=\left(2x^4+1\right)^{\tan x}.
\end{aligned}$$

Example 3.82

Using Logarithmic Differentiation

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$$\begin{aligned}
\ln y &= \ln\frac{x\sqrt{2x+1}}{e^x\sin^3 x} && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= \ln x + \frac{1}{2}\ln(2x+1) - x\ln e - 3\ln\sin x && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x} && \text{Step 3. Diffe entiate both sides.}\\
\frac{dy}{dx} &= y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right) && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= \frac{x\sqrt{2x+1}}{e^x\sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right) && \text{Step 5. Substitute } y=\frac{x\sqrt{2x+1}}{e^x\sin^3 x}.
\end{aligned}$$

Example 3.83

Extending the Power Rule

Find the derivative of $y = x^r$ where r is an arbitrary real number.

Solution

The process is the same as in **Example 3.82**, though with fewer complications.

$$\begin{aligned}
\ln y &= \ln x^r && \text{Step 1. Take the natural logarithm of both sides.}\\
\ln y &= r\ln x && \text{Step 2. Expand using properties of logarithms.}\\
\frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} && \text{Step 3. Diffe entiate both sides.}\\
\frac{dy}{dx} &= y\frac{r}{x} && \text{Step 4. Multiply by } y \text{ on both sides.}\\
\frac{dy}{dx} &= x^r\frac{r}{x} && \text{Step 5. Substitute } y = x^r.\\
\frac{dy}{dx} &= rx^{r-1} && \text{Simplify.}
\end{aligned}$$

3.54 Use logarithmic differentiation to find the derivative of $y = x^x$.

3.55 Find the derivative of $y = (\tan x)^{\pi}$.

Section 3.7: Related Rates

5.1 Related Rates

When defining the derivative $f'(x)$, we define it to be exactly the rate of change of $f(x)$ with respect to x. Consequently, any question about rates of change can be rephrased as a question about derivatives. **When we calculate derivatives, we are calculating rates of change**. Results and answers we obtain for derivatives translate directly into results and answers about rates of change. Let us look at some examples where more than one variable is involved, and where our job is to analyze and exploit relations between the rates of change of these variables. The mathematical step of relating the rates of change turns out to be largely an exercise in differentiation using the chain rule or implicit differentiation. This explains why some textbooks place this section shortly after the sections on the chain rule and implicit differentiation.

Suppose we have two variables x and y (in most problems the letters will be different, but for now let's use x and y) which are both changing with time. A "related rates" problem is a problem in which we know one of the rates of change at a given instant—say, $\dot{x} = dx/dt$—and we want to find the other rate $\dot{y} = dy/dt$ at that instant. (The use of $\dot{x}$ to mean dx/dt goes back to Newton and is still used for this purpose, especially by physicists.)

If y is written in terms of x, i.e., $y = f(x)$, then this is easy to do using the chain rule:

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx}\dot{x}.$$

That is, find the derivative of $f(x)$, plug in the value of x at the instant in question, and multiply by the given value of $\dot{x} = dx/dt$ to get $\dot{y} = dy/dt$.

Example 5.1: Speed at which a Coordinate is Changing

Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the x coordinate is 6 and we measure the speed at which the x coordinate of the object is changing and find that $dx/dt = 3$.
At the same time, how fast is the y coordinate changing?

Solution. Using the chain rule, $dy/dt = 2x \cdot dx/dt$. At $t = 5$ we know that $x = 6$ and $dx/dt = 3$, so $dy/dt = 2 \cdot 6 \cdot 3 = 36$. ♣

In many cases, particularly interesting ones, x and y will be related in some other way, for example $x = f(y)$, or $F(x,y) = k$, or perhaps $F(x,y) = G(x,y)$, where $F(x,y)$ and $G(x,y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, x, y, and $\dot{x}$), and then solving for $\dot{y}$.

To summarize, here are the steps in doing a related rates problem.

Steps for Solving Related Rates Problems

1. Decide what the two variables are.
2. Find an equation relating them.
3. Take d/dt of both sides.
4. Plug in all known values at the instant in question.
5. Solve for the unknown rate.

Example 5.2: Receding Airplanes

A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Solution. To see what's going on, we first draw a schematic representation of the situation, as in Figure 5.1.

Because the plane is in level flight directly away from you, the rate at which x changes is the speed of the plane, $dx/dt = 500$. The distance between you and the plane is y; it is dy/dt that we wish to know. By the Pythagorean Theorem we know that $x^2 + 9 = y^2$. Taking the derivative:

$$2x\dot{x} = 2y\dot{y}.$$

We are interested in the time at which $x = 4$; at this time we know that $4^2 + 9 = y^2$, so $y = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)\dot{y}.$$

Thus, $\dot{y} = 400$ mph. ♣

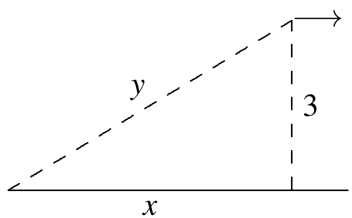


Figure 5.1: Receding airplane.

Example 4.1

Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\text{ cm}^3/\text{sec}$ (**Figure 4.2**). How fast is the radius increasing when the radius is 3 cm?

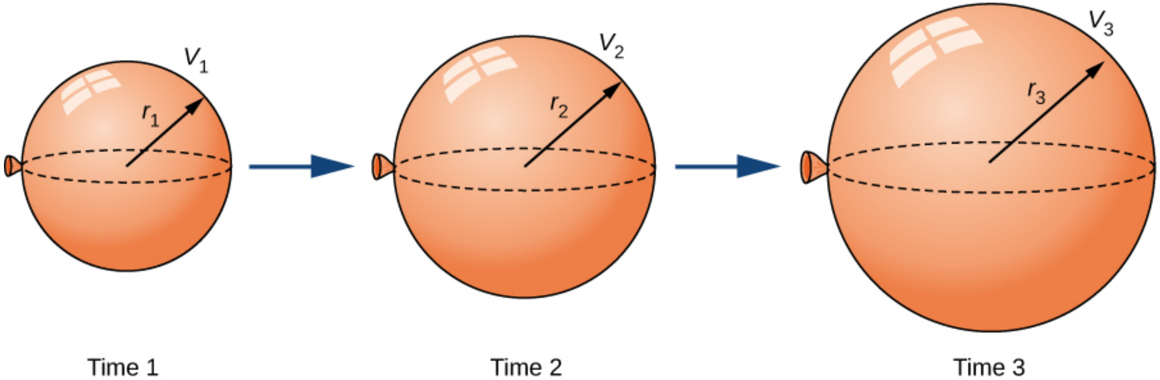


Figure 4.2 As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

Source: OpenStax, Calculus Volume 1, 2019, pg 342

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi[r(t)]^3 \text{ cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi[r(t)]^2 r'(t).$$

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so $V'(t) = 2 \text{ cm}^3/\text{sec}.$ Therefore,

$$2\text{cm}^3/\text{sec} = \left(4\pi[r(t)]^2 \text{ cm}^2\right) \cdot \left(r'(t) \text{cm/s}\right),$$

which implies

$$r'(t) = \frac{1}{2\pi[r(t)]^2} \text{cm/sec}.$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi} \text{cm/sec}.$$

4.1 What is the instantaneous rate of change of the radius when $r = 6 \text{ cm}$?

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example 4.4

Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ \text{ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution

Step 1: Draw a picture introducing the variables.

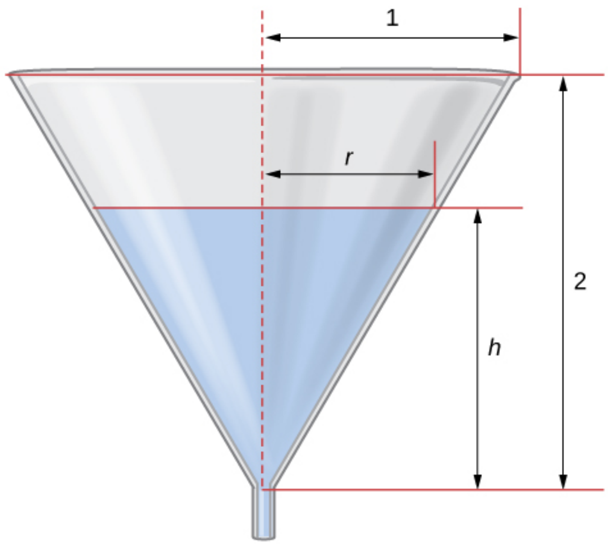


Figure 4.6 Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Source: OpenStax, Calculus Volume 1, 2019, pg 348

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\ \text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\ \text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\ \text{ft/sec}.$$

4.4 At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Example 5.5: Swing Set

A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.

Solution. We start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Note that the person pushing the swing is moving horizontally at a rate we know. In other words, the horizontal coordinate of P is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point of attachment. Then the rate we know is dx/dt, and in part (a) the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $\dot{\theta} = d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram we see that we have a right triangle whose legs are x and $10-y$, and whose hypotenuse is 10. Hence $x^2 + (10-y)^2 = 100$. Taking the derivative of both sides we obtain: $2x\dot{x} + 2(10-y)(0-\dot{y}) = 0$. We now look at what we know after 1 second, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), thus $y = 2$ (because we get $10-y = 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{x} = 6$. Putting in these values gives us $2\cdot 6\cdot 6 - 2\cdot 8\dot{y} = 0$, from which we can easily solve for $\dot{y}$: $\dot{y} = 4.5$ ft/sec.

(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking derivatives we obtain $(\cos\theta)\dot{\theta} = 0.1\dot{x}$. At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6–8–10, $\cos\theta = 8/10$ and $\dot{x} = 6$. Thus $(8/10)\dot{\theta} = 6/10$, i.e., $\dot{\theta} = 6/8 = 3/4$ rad/sec, or approximately 43 deg/sec. ♣

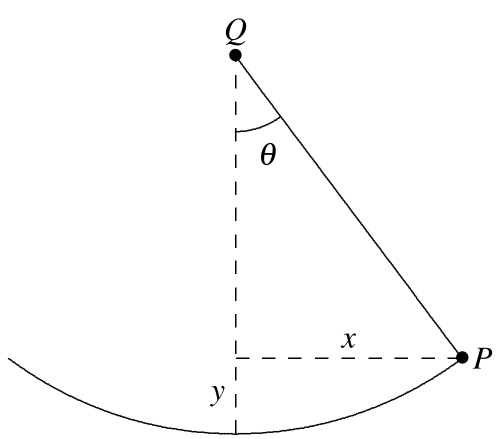


Figure 5.3: Swing.

We have seen that sometimes there are apparently more than two variables that change with time, but in reality there are just two, as the others can be expressed in terms of just two. However sometimes there really are several variables that change with time; as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there are just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

Example 5.6: Distance Changing Rate

A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Solution. Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t. By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$. Taking derivatives we get $2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$, so

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{a\dot{a} + b\dot{b}}{\sqrt{a^2 + b^2}}.$$

Substituting known values we get:

$$\dot{c} = \frac{10 \cdot 80 + 15 \cdot 100}{\sqrt{10^2 + 15^2}} = \frac{460}{\sqrt{13}} \approx 127.6\text{km/hr}$$

at the time of interest. ♣

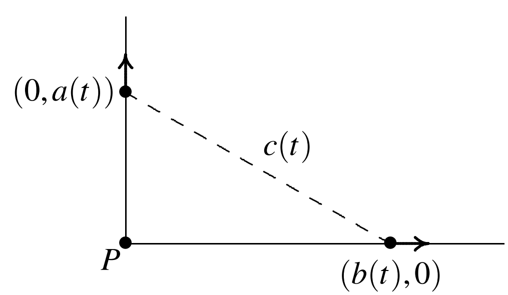


Figure 5.4: Cars moving apart.

Notice how this problem differs from Example 5.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in Example 5.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this Example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Section 3.8: Linear Approximations and Differentials

4.2 | Linear Approximations and Differentials

Learning Objectives

4.2.1 Describe the linear approximation to a function at a point.
4.2.2 Write the linearization of a given function.
4.2.3 Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
4.2.4 Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials **Introduction to Power Series and Functions (http://cnx.org/content/m53760/latest/)** .

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

Figure 4.7(a) shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (**Figure 4.7**(b)). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y-value of the corresponding point on the tangent line is

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.

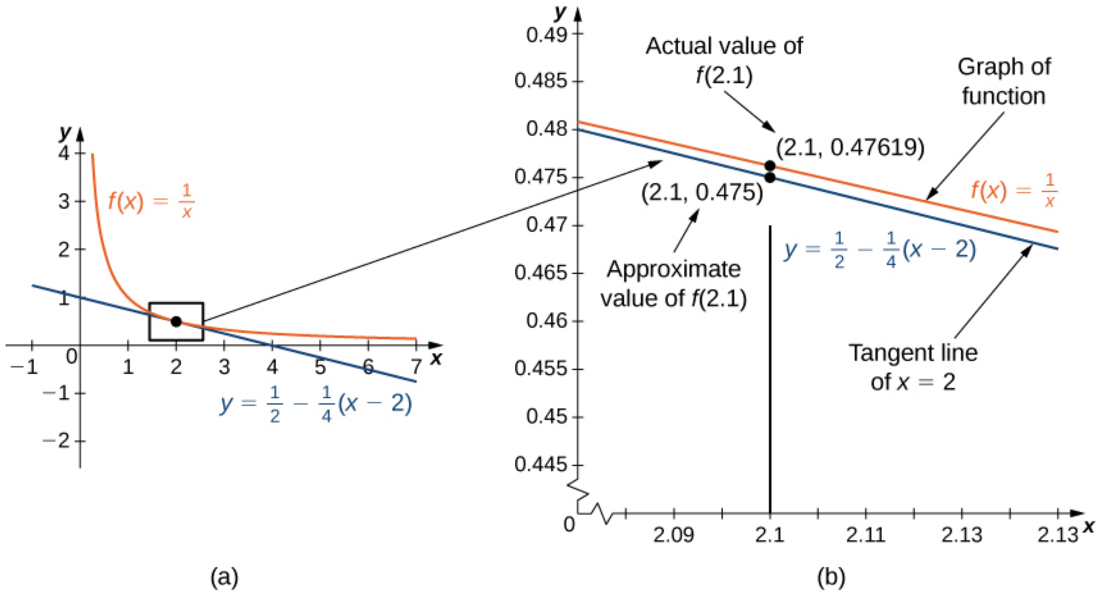


Figure 4.7 (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1,$ the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1,$ which is approximately 0.47619.

In general, for a differentiable function $f,$ the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{4.1}$$

the **linear approximation**, or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example 5.27: Linear Approximation

Let $f(x) = \sqrt{x+4}$, what is $f(6)$?

Solution. We are asked to calculate $f(6) = \sqrt{6+4} = \sqrt{10}$ which is not easy to do without a calculator. However 9 is (relatively) close to 10 and of course $f(5) = \sqrt{9}$ is easy to compute, and we use this to approximate $\sqrt{10}$.

To do so we have $f'(x) = 1/(2\sqrt{x+4})$, and thus the linear approximation to f at $x = 5$ is

$$L(x) = \left(\frac{1}{2\sqrt{5+4}}\right)(x-5) + \sqrt{5+4} = \frac{x-5}{6} + 3.$$

Now to estimate $\sqrt{10}$, we substitute 6 into the linear approximation $L(x)$ instead of $f(x)$, to obtain

$$\sqrt{6+4} \approx \frac{6-5}{6} + 3 = \frac{19}{6} = 3^{1}\!/_{6} = 3.1\bar{6} \approx 3.17$$

It turns out the exact value of $\sqrt{10}$ is actually $3.16227766\ldots$ but our estimate of 3.17 was very easy to obtain and is relatively accurate. This estimate is only accurate to one decimal place. ♣

With modern calculators and computing software it may not appear necessary to use linear approximations, but in fact they are quite useful. For example in cases requiring an explicit numerical approximation, they allow us to get a quick estimate which can be used as a "reality check" on a more complex calculation. Further in some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible without serious loss of accuracy.

Source: Lyryx Learning; Calculus: Early Transcendentals, 2017 , pg 185

Example 5.28: Linear Approximation of Sine

Find the linear approximation of $\sin x$ *at* $x = 0$*, and use it to compute small values of* $\sin x$*.*

Solution. If $f(x) = \sin x$, then $f'(x) = \cos x$, and thus the linear approximation of $\sin x$ at $x = 0$ is:

$$L(x) = \cos(0)(x-0) + \sin(0) = x.$$

Thus when x is small this is quite a good approximation and is used frequently by engineers and scientists to simplify some calculations.

For example you can use your calculator (in radian mode since the derivative of $\sin x$ is $\cos x$ only in radian) to see that

$$\sin(0.1) = 0.099833416\ldots$$

and thus $L(0.1) = 0.1$ is a very good and quick approximation without any calculator! ♣

Example 4.5

Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution

Since we are looking for the linear approximation at $x = 9$, using **Equation 4.1** we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

Source: OpenStax, Calculus Volume 1, 2019, pg 355

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(9) = \sqrt{9} = 3$$
$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Therefore, the linear approximation is given by **Figure 4.8**.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

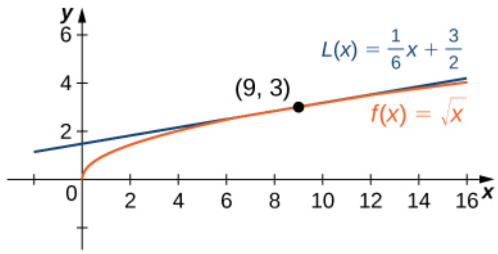


Figure 4.8 The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

4.5 Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Example 4.6

Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62°)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to $60°$, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right).$$

We see that

$$f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by **Figure 4.9**.

$$L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

To estimate $\sin(62°)$ using L, we must first convert $62°$ to radians. We have $62° = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62°)$ is given by

$$\sin(62°) = f\left(\frac{62\pi}{180}\right) \approx L\left(\frac{62\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{62\pi}{180} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{2\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{180} \approx 0.88348.$$

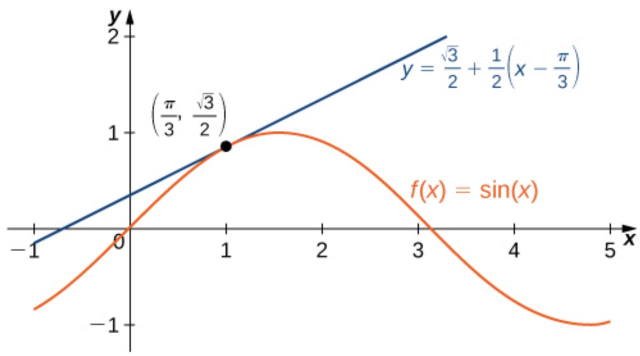


Figure 4.9 The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

4.6 Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m.

Example 4.7

Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1 + x)^n \;\Rightarrow\; f(0) = 1$$
$$f'(x) = n(1 + x)^{n-1} \;\Rightarrow\; f'(0) = n,$$

the linear approximation is given by **Figure 4.10**(a).

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

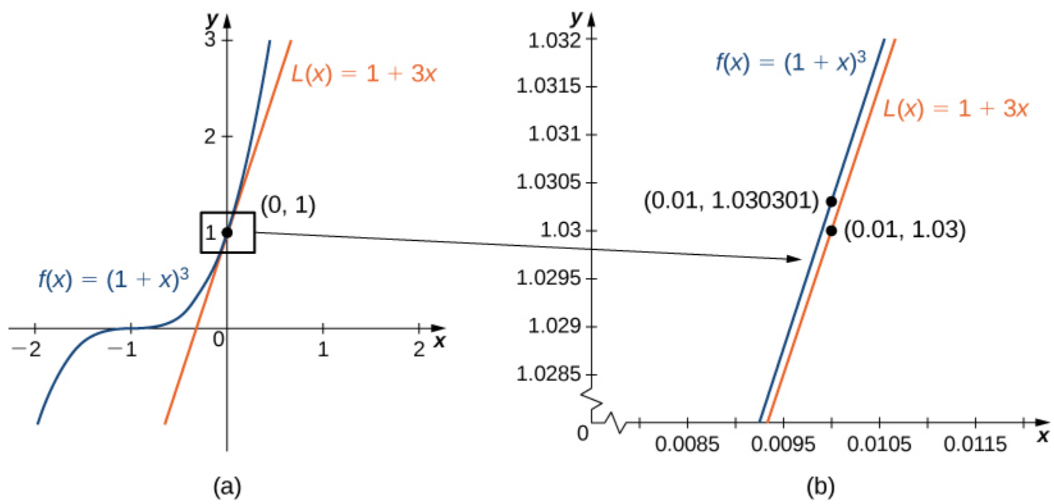


Figure 4.10 (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

4.7 Find the linear approximation of $f(x) = (1 + x)^4$ at $x = 0$ without using the result from the preceding example.

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions *dy* and *dx* in this notation, they did not have meaning on their own. Here we see a meaning to the expressions *dy* and *dx*. Suppose $y = f(x)$ is a differentiable function. Let *dx* be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)dx. \tag{4.2}$$

It is important to notice that dy is a function of both x and dx. The expressions *dy* and *dx* are called *differentials*. We can divide both sides of **Equation 4.2** by dx, which yields

$$\frac{dy}{dx} = f'(x). \tag{4.3}$$

This is the familiar expression we have used to denote a derivative. **Equation 4.2** is known as the **differential form** of **Equation 4.3**.

Example 4.8

Computing differentials

For each of the following functions, find *dy* and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$

b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain *dy* directly.

a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x$, $f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x\, dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1 \sin(3).$$

4.8 For $y = e^{x^2}$, find dy.

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ can be approximated by the linear approximation

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a+dx)-f(a)\approx L(a+dx)-f(a)=f'(a)dx.$$

In other words, the actual change in the function f if x increases from a to $a+dx$ is approximately the difference between $L(a+dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)dx$. In summary,

$$\Delta y=f(a+dx)-f(a)\approx L(a+dx)-f(a)=f'(a)dx=dy.$$

Therefore, we can use the differential $dy=f'(a)dx$ to approximate the change in y if x increases from $x=a$ to $x=a+dx$. We can see this in the following graph.

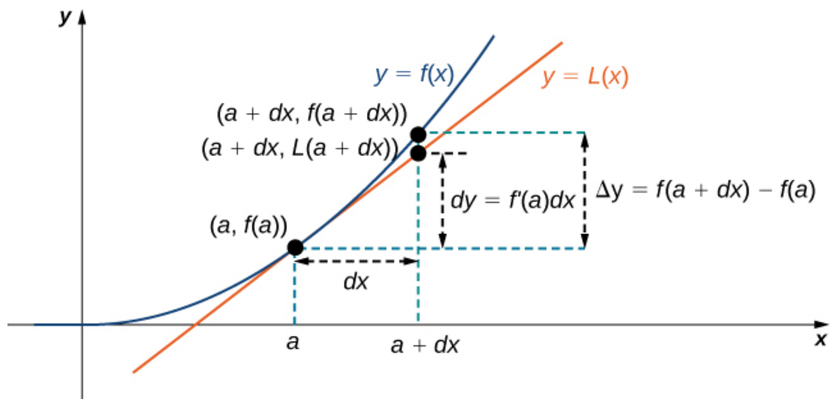


Figure 4.11 The differential $dy=f'(a)dx$ is used to approximate the actual change in y if x increases from a to $a+dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 4.9

Approximating Change with Differentials

Let $y=x^2+2x$. Compute Δy and dy at $x=3$ if $dx=0.1$.

Solution

The actual change in y if x changes from $x=3$ to $x=3.1$ is given by

$$\Delta y=f(3.1)-f(3)=[(3.1)^2+2(3.1)]-[3^2+2(3)]=0.81.$$

The approximate change in y is given by $dy=f'(3)dx$. Since $f'(x)=2x+2$, we have

$$dy=f'(3)dx=(2(3)+2)(0.1)=0.8.$$

4.9 For $y=x^2+2x$, find Δy and dy at $x=3$ if $dx=0.2$.

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a, the propagated error is

$$\Delta y \approx dy = f'(a)dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a + dx$, and estimate

$$\Delta y \approx dy \approx f'(a + dx)dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 4.10

Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \leq dx \leq 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \leq dV \leq 3(5)^2(0.1).$$

Therefore,

$$-7.5 \leq dV \leq 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649 \text{ cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651\text{ cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125.$ Therefore, the error in the computed volume is

$$117.649 - 125 \le \Delta V \le 132.651 - 125.$$

That is,

$$-7.351 \le \Delta V \le 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

4.10 Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

The measurement error $dx\ (=\Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q},$ where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016,$ or $1.6\%.$ By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32},$ or $3.1\%.$ Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 4.11

Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution

If the measurement of the radius is accurate to within $\pm 80,$ we have

$$-80 \le dr \le 80.$$

Since the volume of a sphere is given by $V = \left(\frac{4}{3}\right)\pi r^3,$ we have

$$dV = 4\pi r^2\,dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \le dV \le 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx \left(\frac{4}{3}\right)\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \le \frac{dV}{V} \le \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \le \frac{dV}{V} \le 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

4.11 Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Section 3.9: Hyperbolic Functions

This section will be different from the ones that precede it. We were unable to find an open source textbook that explains this concept in terms of calculus, so we are presenting you with:

1. A definition of the hyperbolic functions from the LyryxLearning textbook
2. A video that ties this video to the Calculus concepts that have previously been covered
3. A series of video examples to illustrate the topic (featured on this page, but should be watched after the section has been read)

This video ties the definition of hyperbolic functions to the Calculus concepts we have been studying:
Professor Dave Explains - Hyperbolic Functions: Definition, Identities, Derivatives, and Inverses

Examples:

This video explains hyperbolic identities by proving one of them: Professor Dave Explains - Hyperbolic Functions: Definition, Identities, Derivatives, and Inverses

This video ties hyperbolic differentiation with the Chain Rule: Mathispower4u - Ex 2: Derivatives of Hyperbolic Functions with the Chain Rule

This video explains the expression of hyperbolic functions in terms of logarithms: blackpenredpen - inverse sinh(x)

This video provides more examples of hyperbolic differentiation: patrickJMT - Hyperbolic Functions - Derivatives

2.7 Hyperbolic Functions

Source: Lyryx Learning, Calculus: Early Transcendentals, 2017, pg 68

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

Definition 2.29: Hyperbolic Sine and Cosine

The **hyperbolic cosine** *is the function*

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the **hyperbolic sine** *is the function*

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Notice that cosh is even (that is, $\cosh(-x) = \cosh(x)$) while sinh is odd ($\sinh(-x) = -\sinh(x)$), and $\cosh x + \sinh x = e^x$. Also, for all x, $\cosh x > 0$, while $\sinh x = 0$ if and only if $e^x - e^{-x} = 0$, which is true precisely when $x = 0$.

Theorem 2.30: Range of Hyperbolic Cosine

The range of $\cosh x$ *is* $[1, \infty)$.

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} \\ 2y &= e^x + e^{-x} \\ 2ye^x &= e^{2x} + 1 \\ 0 &= e^{2x} - 2ye^x + 1 \\ e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1} \end{aligned}$$

From the last equation, we see $y^2 \geq 1$, and since $y \geq 0$, it follows that $y \geq 1$.

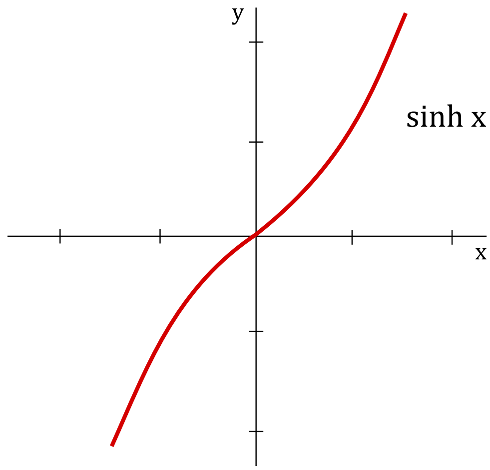
Now suppose $y \geq 1$, so $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number, and $y = \cosh x$, so y is in the range of $\cosh(x)$. ♣

Definition 2.31: Hyperbolic Functions

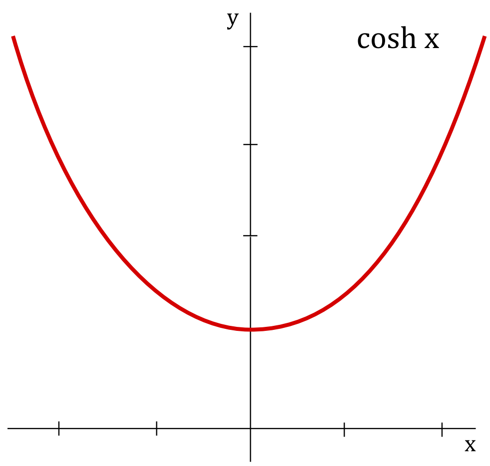
We can also define hyperbolic functions for the other trigonometric functions as you would expect:

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

The graph of $\sinh x$ is shown below:



The graph of $\cosh x$ is shown below:



Example 2.32: Computing Hyperbolic Tangent

Compute $\tanh(\ln 2)$.

Solution. This uses the definitions of the hyperbolic functions.

$$\tanh(\ln 2) = \frac{\sinh(\ln 2)}{\cosh(\ln 2)} = \frac{\dfrac{e^{\ln 2} - e^{-\ln 2}}{2}}{\dfrac{e^{\ln 2} + e^{-\ln 2}}{2}} = \frac{\dfrac{2-(1/2)}{2}}{\dfrac{2+(1/2)}{2}} = \frac{2-(1/2)}{2+(1/2)} = \frac{3}{5}$$

♣

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

Theorem 2.33: Hyperbolic Identity

For all x in $\mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$.

Proof. The proof is a straightforward computation:

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.$$

♣

This immediately gives two additional identities:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \text{and} \qquad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose x-intercepts are ± 1. If (x, y) is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm\sqrt{x^2 - 1} = \pm\sqrt{\cosh^2 x - 1} = \pm \sinh t$. So for some suitable t, $\cosh t$ and $\sinh t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that t is twice the area shown in the first graph of figure 2.4. Even this is analogous to trigonometry; $\cos t$ and $\sin t$ are the coordinates of a typical point on the unit circle, and t is twice the area shown in the second graph of Figure 2.4.

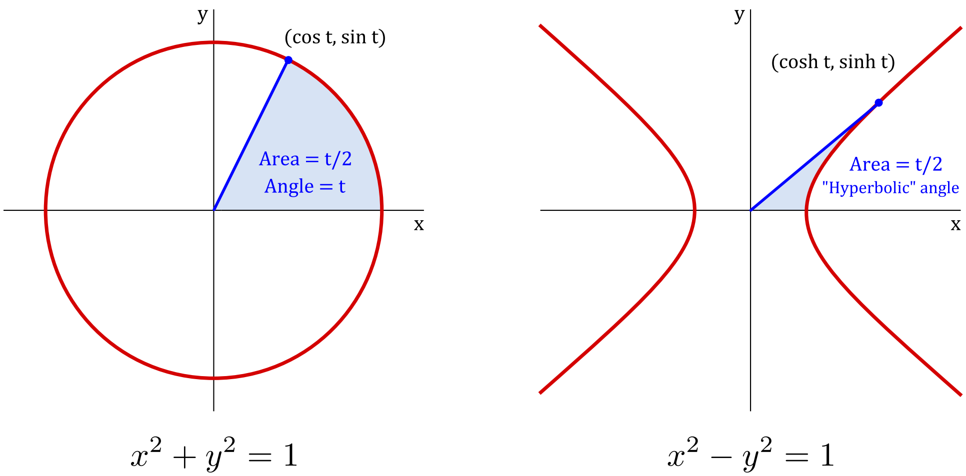


Figure 2.4: Geometric definitions. Here, t is twice the shaded area in each figure.

Since $\cosh x > 0$, $\sinh x$ is increasing and hence one-to-one, so $\sinh x$ has an inverse, $\operatorname{arcsinh} x$. Also, $\sinh x > 0$ when $x > 0$, so $\cosh x$ is injective on $[0, \infty)$ and has a (partial) inverse, $\operatorname{arccosh} x$. The other hyperbolic functions have inverses as well, though $\operatorname{arcsech} x$ is only a partial inverse.

Chapter 4: Applications of Differentiation

Section 4.1: Maximum and Minimum Values

5.2 Extrema of a Function

In calculus, there is much emphasis placed on analyzing the behaviour of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? How does the interval I impact our discussion of extrema?

5.2.1. Local Extrema

A **local maximum** point on a function is a point (x,y) on the graph of the function whose y coordinate is larger than all other y coordinates on the graph at points "close to" (x,y). More precisely, $(x,f(x))$ is a local maximum if there is an interval (a,b) with $a < x < b$ and $f(x) \geq f(z)$ for every z in (a,b). Similarly, (x,y) is a **local minimum** point if it has locally the smallest y coordinate. Again being more precise: $(x,f(x))$ is a local minimum if there is an interval (a,b) with $a < x < b$ and $f(x) \leq f(z)$ for every z in (a,b). A **local extremum** is either a local minimum or a local maximum.

Definition 5.7: Local Maxima and Minima

A real-valued function f has a **local maximum** *at x_0 if $f(x_0)$ is the largest value of f near x_0; in other words, $f(x_0 \geq f(x)$ when x is near x_0.*

A real-valued function f has a **local minimum** *at x_0 if $f(x_0)$ is the smallest value of f near x_0; in other words, $f(x_0) \leq f(x)$ when x is near x_0.*

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in Figure 5.5.

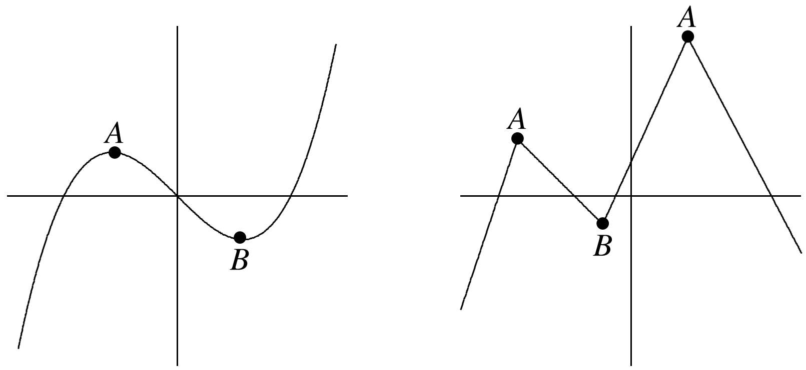


Figure 5.5: Some local maximum points (A) and minimum points (B).

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line *must* be horizontal. This is important enough to state as a theorem.

The proof is simple enough and we include it here, but you may accept Fermat's Theorem based on its strong intuitive appeal and come back to its proof at a later time.

Theorem 5.8: Fermat's Theorem

If $f(x)$ has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.

Proof. We shall give the proof for the case where $f(x)$ has a local maximum at $x = a$. The proof for the local minimum case is similar.

Since $f(x)$ has a local maximum at $x = a$, there is an open interval (c, d) with $c < a < d$ and $f(x) \leq f(a)$ for every x in (c, d). So, $f(x) - f(a) \leq 0$ for all such x. Let us now look at the sign of the difference quotient $\frac{f(x) - f(a)}{x - a}$. We consider two cases according as $x > a$ or $x < a$.

If $x > a$, then $x - a > 0$ and so, $\frac{f(x) - f(a)}{x - a} \leq 0$. Taking limit as x approach a from the right, we get

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

On the other hand, if $x < a$, then $x - a < 0$ and so, $\frac{f(x) - f(a)}{x - a} \geq 0$. Taking limit as x approach a from the left, we get

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0.$$

Since f is differentiable at a, $f'(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$. Therefore, we have both $f'(a) \leq 0$ and $f'(a) \geq 0$. So, $f'(a) = 0$. ♣

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, as in the left hand graph in Figure 5.5, or the derivative is undefined, as in the right hand graph. Any value of x in the domain of f for which $f'(x)$ is zero or undefined is called a **critical**

point for f. When looking for local maximum and minimum points, you are likely to make two sorts of mistakes: You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere. You might also assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true. A portion of the graph of $f(x) = x^3$ is shown in Figure 5.6. The derivative of f is $f'(x) = 3x^2$, and $f'(0) = 0$, but there is neither a maximum nor minimum at $(0,0)$.

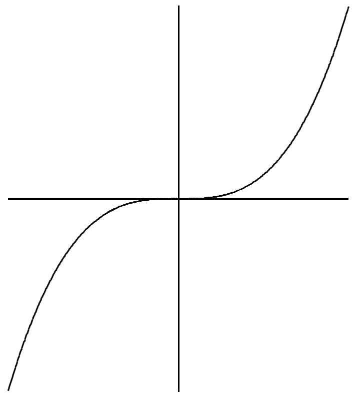


Figure 5.6: No maximum or minimum even though the derivative is zero.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the y coordinates "near" the potential maximum or minimum are above or below the y coordinate at the point of interest. Of course, there are too many points "near" the point to test, but a little thought shows we need only test two provided we know that f is continuous (recall that this means that the graph of f has no jumps or gaps).

Suppose, for example, that we have identified three points at which f' is zero or nonexistent: (x_1,y_1), (x_2,y_2), (x_3,y_3), and $x_1 < x_2 < x_3$ (see Figure 5.7). Suppose that we compute the value of $f(a)$ for $x_1 < a < x_2$, and that $f(a) < f(x_2)$. What can we say about the graph between a and x_2? Could there be a point $(b, f(b))$, $a < b < x_2$ with $f(b) > f(x_2)$? No: if there were, the graph would go up from $(a, f(a))$ to $(b, f(b))$ then down to $(x_2, f(x_2))$ and somewhere in between would have a local maximum point. (This is not obvious; it is a result of the Extreme Value Theorem.) But at that local maximum point the derivative of f would be zero or nonexistent, yet we already know that the derivative is zero or nonexistent only at x_1, x_2, and x_3. The upshot is that one computation tells us that $(x_2, f(x_2))$ has the largest y coordinate of any point on the graph near x_2 and to the left of x_2. We can perform the same test on the right. If we find that on both sides of x_2 the values are smaller, then there must be a local maximum at $(x_2, f(x_2))$; if we find that on both sides of x_2 the values are larger, then there must be a local minimum at $(x_2, f(x_2))$; if we find one of each, then there is neither a local maximum or minimum at x_2.

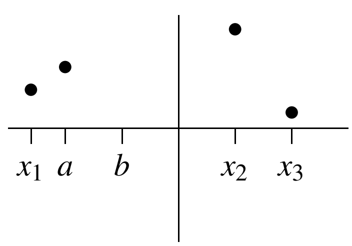


Figure 5.7: Testing for a maximum or minimum.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

Example 5.9: Simple Cubic

Find all local maximum and minimum points for the function $f(x) = x^3 - x$.

Solution. The derivative is $f'(x) = 3x^2 - 1$. This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that $f(\sqrt{3}/3) = -2\sqrt{3}/9$. Now we test two points on either side of $x = \sqrt{3}/3$, choosing one point in the interval $(-\sqrt{3}/3, \sqrt{3}/3)$ and one point in the interval $(\sqrt{3}/3, \infty)$. Since $f(0) = 0 > -2\sqrt{3}/9$ and $f(1) = 0 > -2\sqrt{3}/9$, there must be a local minimum at $x = \sqrt{3}/3$. For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$. ♣

Of course this example is made very simple by our choice of points to test, namely $x = -1$, 0, 1. We could have used other values, say $-5/4$, $1/3$, and $3/4$, but this would have made the calculations considerably more tedious, and we should always choose very simple points to test if we can.

Example 5.10: Max and Min

Find all local maximum and minimum points for $f(x) = \sin x + \cos x$.

Solution. The derivative is $f'(x) = \cos x - \sin x$. This is always defined and is zero whenever $\cos x = \sin x$. Recalling that the $\cos x$ and $\sin x$ are the x and y coordinates of points on a unit circle, we see that $\cos x = \sin x$ when x is $\pi/4$, $\pi/4 \pm \pi$, $\pi/4 \pm 2\pi$, $\pi/4 \pm 3\pi$, etc. Since both sine and cosine have a period of 2π, we need only determine the status of $x = \pi/4$ and $x = 5\pi/4$. We can use 0 and $\pi/2$ to test the critical value $x = \pi/4$. We find that $f(\pi/4) = \sqrt{2}$, $f(0) = 1 < \sqrt{2}$ and $f(\pi/2) = 1$, so there is a local maximum when $x = \pi/4$ and also when $x = \pi/4 \pm 2\pi$, $\pi/4 \pm 4\pi$, etc. We can summarize this more neatly by saying that there are local maxima at $\pi/4 \pm 2k\pi$ for every integer k.

We use π and 2π to test the critical value $x = 5\pi/4$. The relevant values are $f(5\pi/4) = -\sqrt{2}$, $f(\pi) = -1 > -\sqrt{2}$, $f(2\pi) = 1 > -\sqrt{2}$, so there is a local minimum at $x = 5\pi/4$, $5\pi/4 \pm 2\pi$, $5\pi/4 \pm 4\pi$, etc. More succinctly, there are local minima at $5\pi/4 \pm 2k\pi$ for every integer k. ♣

Example 5.11: Max and Min

Find all local maximum and minimum points for $g(x) = x^{2/3}$.

Solution. The derivative is $g'(x) = \frac{2}{3}x^{-1/3}$. This is undefined when $x = 0$ and is not equal to zero for any x in the domain of g'. Now we test two points on either side of $x = 0$. We use $x = -1$ and $x = 1$. Since $g(0) = 0$, $g(-1) = 1 > 0$ and $g(1) = 1 > 0$, there must be a local minimum at $x = 0$. ♣

Additional Example of Local Extrema:

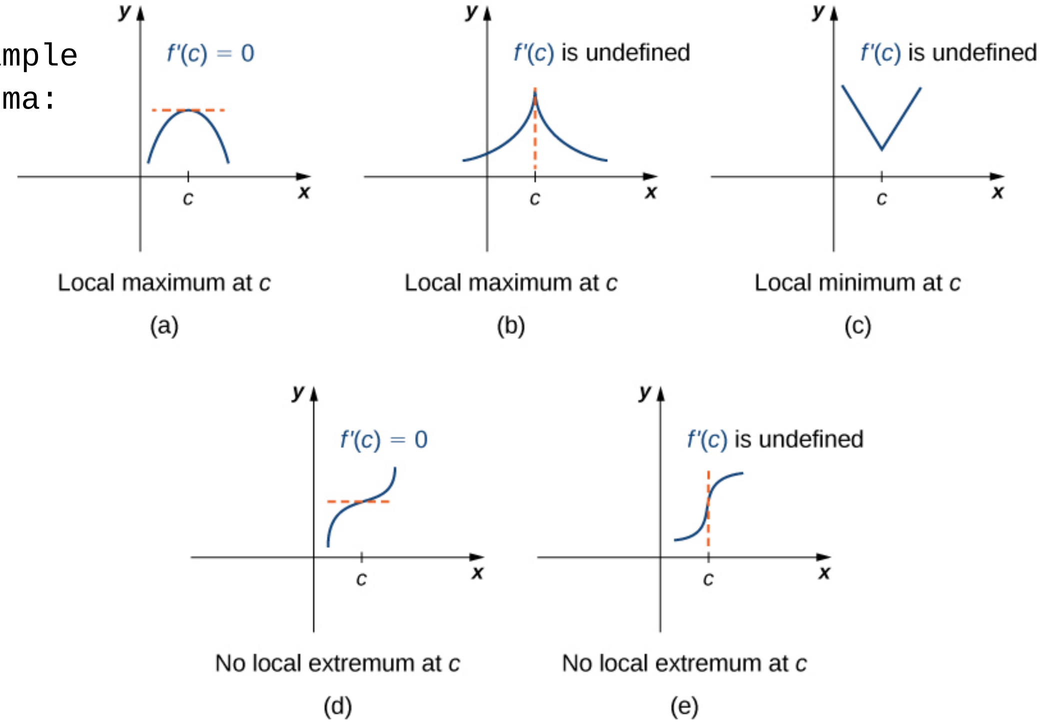


Figure 4.15 (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 4.12

Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

b. $f(x) = \left(x^2 - 1\right)^3$

c. $f(x) = \frac{4x}{1 + x^2}$

Solution

Source: OpenStax; Calculus Volume 1; 2019, pg 370

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in **Figure 4.16**, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

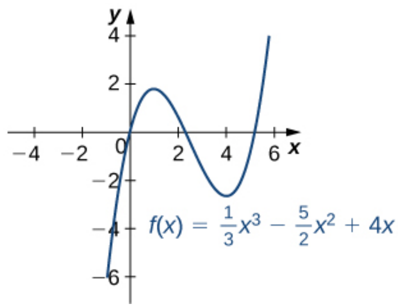


Figure 4.16 This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3\left(x^2 - 1\right)^2 (2x) = 6x\left(x^2 - 1\right)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in **Figure 4.17**, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

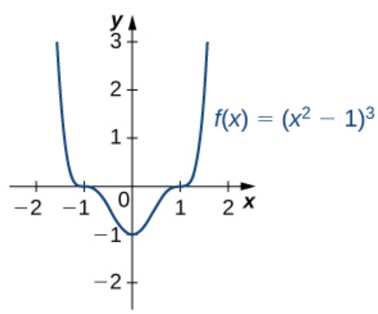


Figure 4.17 This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the chain rule, we see that the derivative is

$$f'(x) = \frac{\left(1 + x^2 4\right) - 4x(2x)}{\left(1 + x^2\right)^2} = \frac{4 - 4x^2}{\left(1 + x^2\right)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in **Figure 4.18**, we see that f has an absolute maximum at $x = 1$

and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

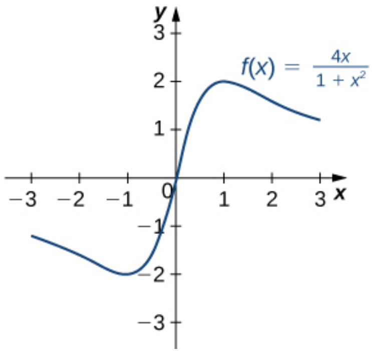


Figure 4.18 This function has an absolute maximum and an absolute minimum.

4.12 Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

5.2.2. Absolute Extrema

Absolute extrema are also commonly referred to as **global extrema**. Unlike local extrema, which are only “extreme" relative to points “close to” them, an absolute (or global) extrema is “extreme" relative to *all* other points in the interval under consideration.

Definition 5.12: Absolute Maxima and Minima

A real-valued function f has an **absolute maximum** *on an interval I at x_0 if $f(x_0)$ is the largest value of f on I; in other words, $f(x_0) \geq f(x)$ for all x in the domain of f that are in I.*

A real-valued function f has an **absolute minimum** *on an interval I at x_0 if $f(x_0)$ is the smallest value of f on I; in other words, $f(x_0) \leq f(x)$ for all x in the domain of f that are in I.*

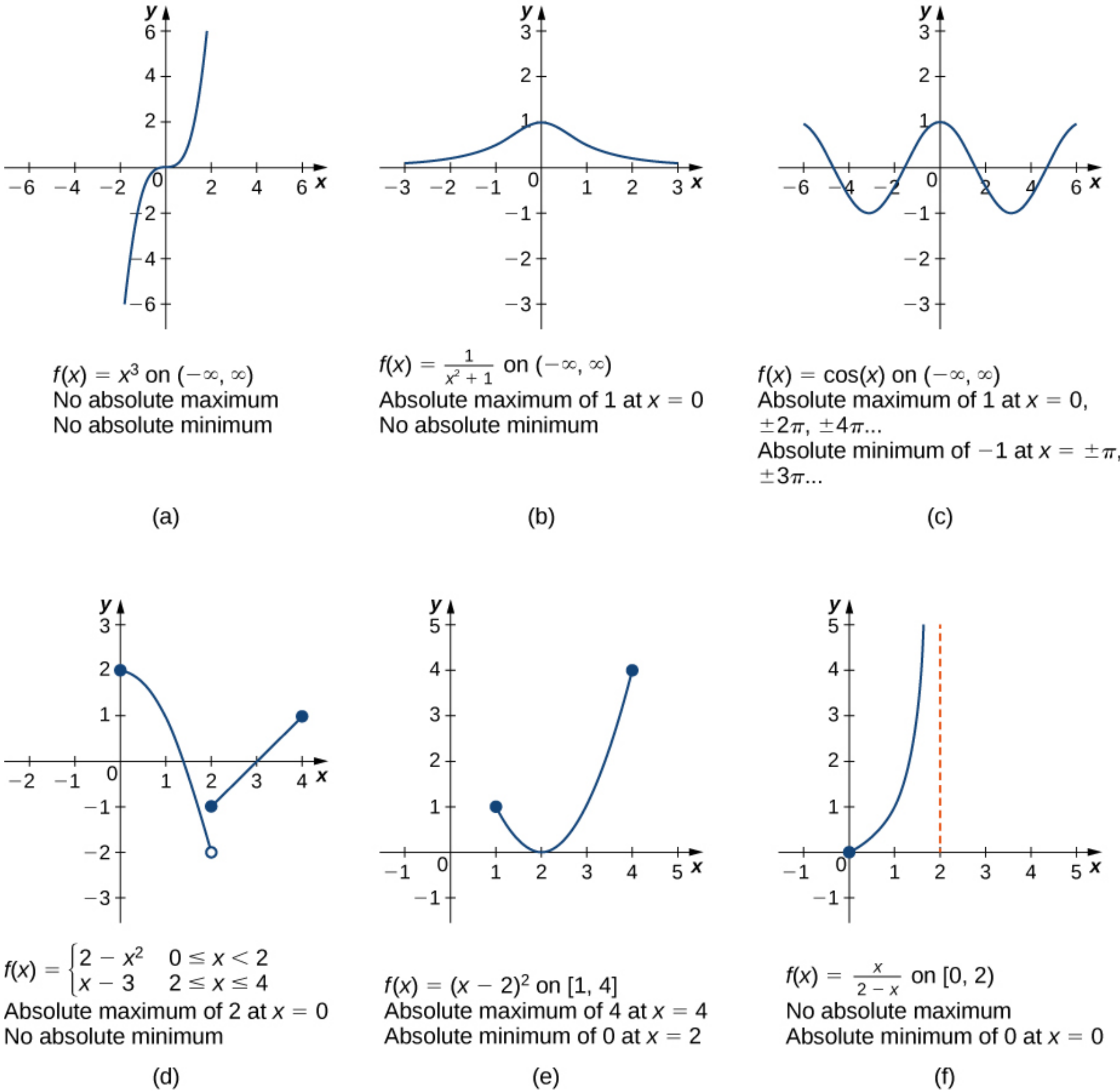


Figure 4.13 Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Like Fermat's Theorem, the following theorem has an intuitive appeal. However, unlike Fermat's Theorem, the proof relies on a more advanced concept called **compactness**, which will only be covered in a course typically entitled Analysis. So, we will be content with understanding the statement of the theorem.

Theorem 5.14: Extreme-Value Theorem

If a function f is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum on $[a,b]$.

Although this theorem tells us that an absolute extremum exists, it does not tell us what it is or how to find it.

Note that if an absolute extremum is inside the interval (i.e. not an endpoint), then it must also be a local extremum. This immediately tells us that to find the absolute extrema of a function on an interval, we need only examine the local extrema inside the interval, and the endpoints of the interval.

We can devise a method for finding absolute extrema for a function f on a closed interval $[a,b]$:

1. Verify the function is continuous on $[a,b]$.

2. Find the derivative and determine all critical values of f that are in $[a,b]$.

3. Evaluate the function at the critical values found in Step 2 and the end points of the interval.

4. Identify the absolute extrema.

Why must a function be continuous on a closed interval in order to use this theorem? Consider the following example.

Example 5.15: Absolute Extrema of a $1/x$

Find any absolute extrema for $f(x) = 1/x$ on the interval $[-1,1]$.

Solution. The function f is not continuous at $x=0$. Since $0 \in [-1,1]$, f is not continous on the closed interval:

$$\lim_{x \to 0^+} f(x) = +\infty$$
$$\lim_{x \to 0^-} f(x) = -\infty,$$

so we are *unable* to apply the Extreme-Value Theorem. Therefore, $f(x) = 1/x$ does not have an absolute maximum or an absolute minimum on $[-1,1]$. ♣

However, if we consider the same function on an interval where it is continuous, the theorem will apply. This is illustrated in the following example.

Example 5.16: Absolute Extrema of a $1/x$

Find any absolute extrema for $f(x) = 1/x$ on the interval $[1,2]$.

Solution. The function f is continous on the interval, so we can apply the Extreme-Value Theorem. We begin with taking the derivative to be $f'(x) = -1/x^2$ which has a critical value at $x=0$, but since this critical value is not in $[1,2]$ we ignore it. The only points where an extrema can occur are the endpoints of the interval. To find the maximum or minimum we can simply evaluate the function: $f(1) = 1$ and $f(2) = 1/2$, so the absolute maximum is at $x=1$ and the absolute minimum is at $x=2$. ♣

Why must an interval be closed in order to use the above theorem? Recall the difference between open and closed intervals. Consider a function f on the open interval $(0,1)$. If we choose successive values of x moving closer and closer to 1, what happens? Since 1 is not included in the interval we will not attain exactly the value of 1. Suppose we reach a value of 0.9999 — is it possible to get closer to 1? Yes: There are infinitely many real numbers between 0.9999 and 1. In fact, any conceivable real number close to 1 will have infinitely many real numbers between itself and 1. Now, suppose f is decreasing on $(0,1)$: As we approach 1, f will continue to decrease, even if the difference between successive values of f is slight. Similarly if f is increasing on $(0,1)$.

Consider a few more examples:

Example 5.17: Determining Absolute Extrema

Determine the absolute extrema of $f(x) = x^3 - x^2 + 1$ on the interval $[-1,2]$.

Solution. First, notice f is continuous on the closed interval $[-1,2]$, so we're able to use Theorem 5.14 to determine the absolute extrema. The derivative is $f'(x) = 3x^2 - 2x$, and the critical values are $x = 0, 2/3$ which are both in the interval $[-1,2]$. In order to find the absolute extrema, we must consider all critical values that lie within the interval (that is, in $(-1,2)$) *and* the endpoints of the interval.

$$\begin{aligned} f(-1) &= (-1)^3 - (-1)^2 + 1 = -1 \\ f(0) &= (0)^3 - (0)^2 + 1 = 1 \\ f(2/3) &= (2/3)^3 - (2/3)^2 + 1 = 23/27 \\ f(2) &= (2)^3 - (2)^2 + 1 = 5 \end{aligned}$$

The absolute maximum is at (2,5) and the absolute minimum is at (-1,-1). ♣

Example 5.18: Determining Absolute Extrema

Determine the absolute extrema of $f(x) = -9/x - x + 10$ on the interval $[2,6]$.

Solution. First, notice f is continuous on the closed interval $[2,6]$, so we're able to use Theorem 5.14 to determine the absolute extrema. The function is not continuous at $x = 0$, but we can ignore this fact since 0 is not in $[2,6]$. The derivative is $f'(x) = 9/x^2 - 1$, and the critical values are $x = \pm 3$, but only $x = +3$ is in the interval. In order to find the absolute extrema, we must consider all critical values that lie within the interval *and* the endpoints of the interval.

$$\begin{aligned} f(2) &= -9/(2) - (2) + 10 = 7/2 = 3.5 \\ f(3) &= -9/(3) - (3) + 10 = 4 \\ f(6) &= -9/(6) - (6) + 10 = 5/2 = 2.5 \end{aligned}$$

The absolute maximum is at (3,4) and the absolute minimum is at (6,2.5). ♣

When we are trying to find the absolute extrema of a function on an open interval, we cannot use the Extreme Value Theorem. However, if the function is continuous on the interval, many of the same ideas apply. In particular, if an absolute extremum exists, it must also be a local extremum. In addition to checking values at the local extrema, we must check the behaviour of the function as it approaches the ends of the interval.

Some examples to illustrate this method.

Example 5.19: Extrema of Secant

Find the extrema of $\sec(x)$ on $(-\pi/2, \pi/2)$.

Solution. Notice $\sec(x)$ is continuous on $(-\pi/2, \pi/2)$ and has one local minimum at 0. Also

$$\lim_{x \to (-\pi/2)^+} \sec(x) = \lim_{x \to (\pi/2)^-} \sec(x) = +\infty,$$

so $\sec(x)$ has no absolute maximum, but the point $(0,1)$ is the absolute minimum. ♣

A similar approach can be used for infinite intervals.

Example 5.20: Extrema of $\frac{x^2}{x^2+1}$

Find the extrema of $\dfrac{x^2}{x^2+1}$ *on* $(-\infty,\infty)$.

Solution. Since $x^2+1 \neq 0$ for all x in $(-\infty,\infty)$ the function is continuous on this interval. This function has only one critical value at $x=0$, which is the local minimum and also the absolute minimum. Now, $\lim_{x\to\pm\infty}\frac{x^2}{x^2+1}=1$, so the function does not have an absolute maximum: It continues to increase towards 1, but does not attain this exact value. ♣

Additional example/explanation of absolute extrema

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in **Figure 4.13**, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by **Fermat's Theorem**, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 4.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From **Location of Absolute Extrema**, the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Source: OpenStax; Calculus Volume 1; 2019 , pg 372

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 4.13

Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.

b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f\left(\frac{3}{2}\right)$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f\left(\frac{3}{2}\right)$ and find

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
0	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval [1, 3] is -2, and it occurs at $x = 3$ as shown in the following graph.

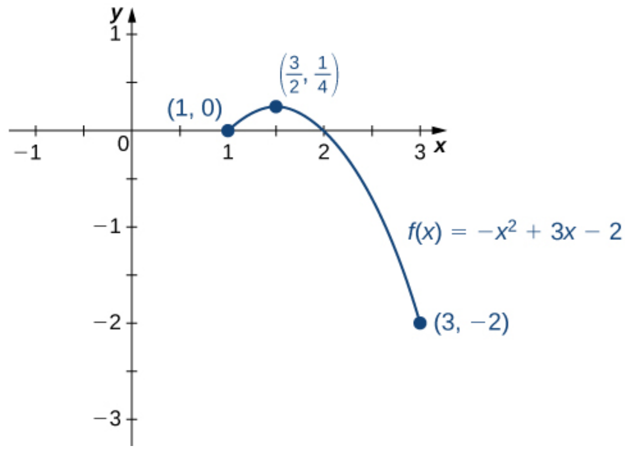


Figure 4.19 This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\sqrt[3]{4} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	−2	Absolute minimum
2	−0.762	

We conclude that the absolute maximum of f over the interval [0, 2] is zero, and it occurs at $x = 0$. The absolute minimum is −2, and it occurs at $x = 1$ as shown in the following graph.

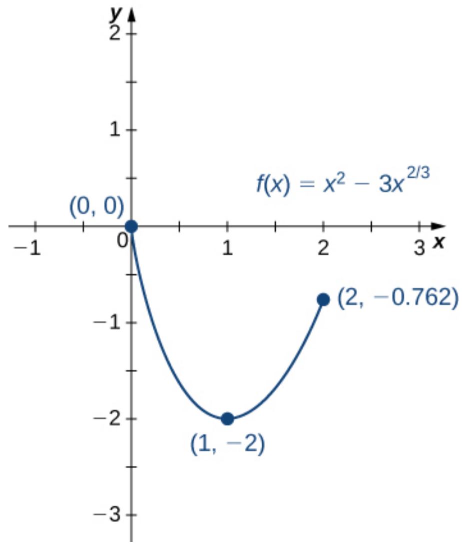


Figure 4.20 This function has an absolute maximum at an endpoint of the interval.

4.13 Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Section 4.2: The Mean Value Theorem

The following videos provide useful examples of Applications to the MVT and Rolle's theorem:
Anil Kumar - Prove Trigonometric Identity from Mean Value Theorem Calculus 2
Anil Kumar - Rolle's Theorem to Prove Exactly one root for Cubic Function AP Calculus

4.4 | The Mean Value Theorem

Learning Objectives

- 4.4.1 Explain the meaning of Rolle's theorem.
- 4.4.2 Describe the significance of the Mean Value Theorem.
- 4.4.3 State three important consequences of the Mean Value Theorem.

The **Mean Value Theorem** is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure 4.21 illustrates this theorem.

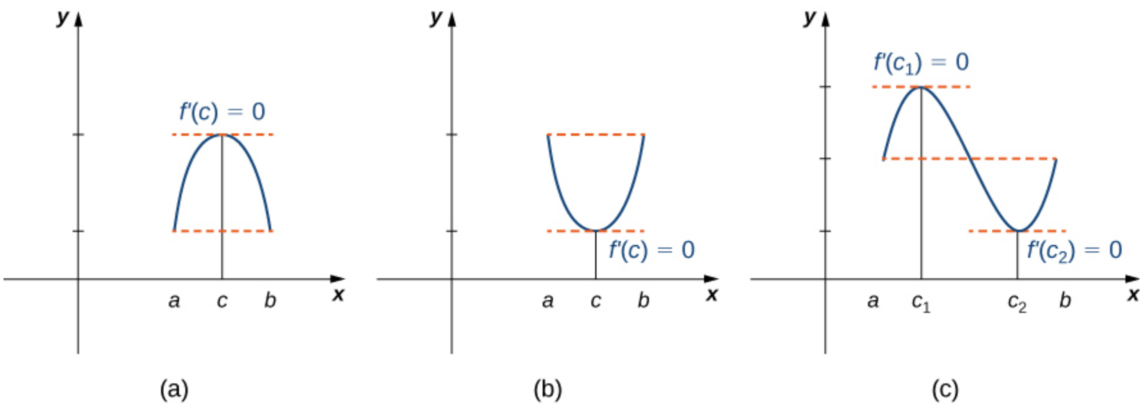


Figure 4.21 If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Theorem 4.4: Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = 0$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1),$ but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

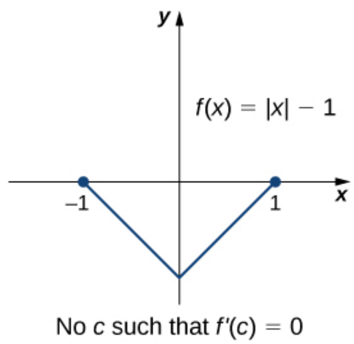


Figure 4.22 Since $f(x) = |x| - 1$ is not differentiable at $x = 0,$ the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0.$

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0.$

Example 4.14

Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0.$

a. $f(x) = x^2 + 2x$ over $[-2, 0]$

b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0).$ Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0.$ Since $f'(x) = 2x + 2 = 2(x + 1),$ we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

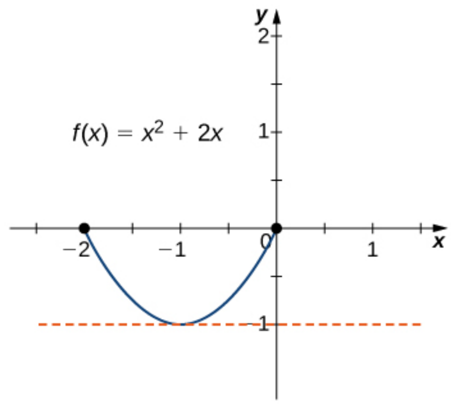


Figure 4.23 This function is continuous and differentiable over $[-2, 0]$, $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

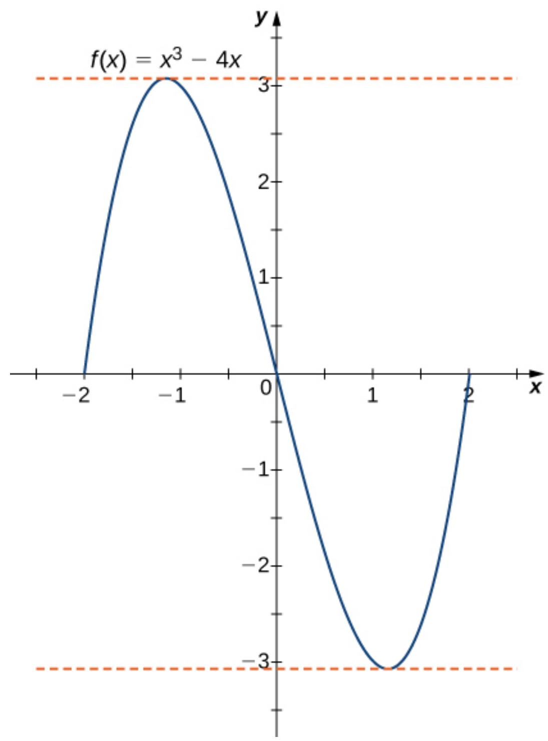


Figure 4.24 For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

4.14 Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (**Figure 4.25**). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

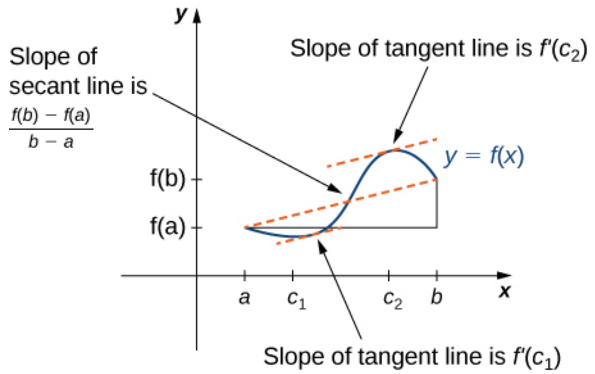


Figure 4.25 The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Theorem 4.5: Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

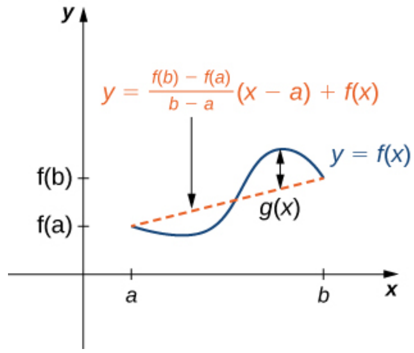


Figure 4.26 The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 4.15

Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (**Figure 4.27**). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

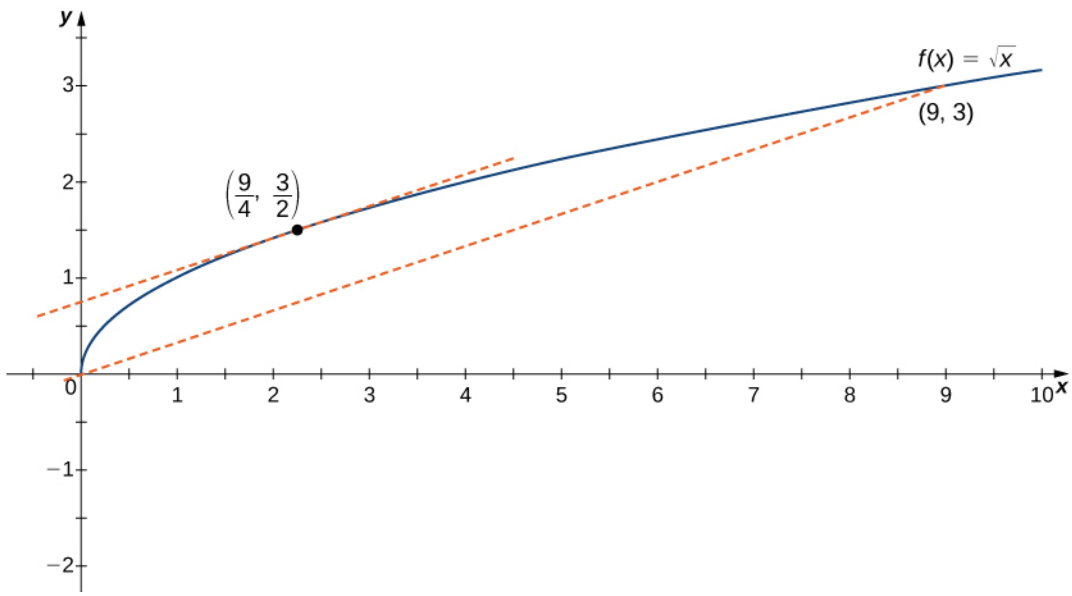


Figure 4.27 The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting $(0, 0)$ and $(9, 3)$.

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \le t \le 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 4.16

Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100.$

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t, we find that $t = \pm\frac{5}{2}$sec. Since we are only considering $t \geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.
b. The average velocity is given by

$$v_{\text{avg}} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{1 - 100}{5/2} = -40 \text{ ft/sec.}$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{\text{avg}} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c, we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

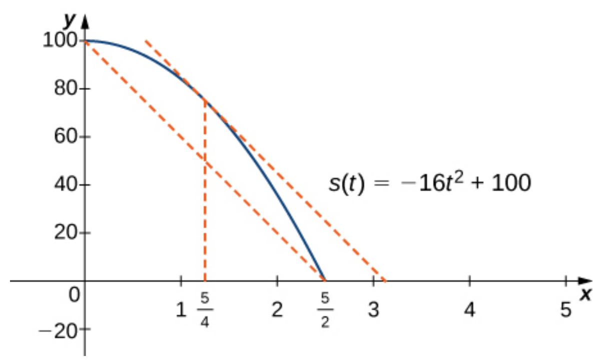


Figure 4.28 At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

4.15 Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Theorem 4.6: Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From **Corollary 1: Functions with a Derivative of Zero**, it follows that if two functions have the same derivative, they differ by, at most, a constant.

Theorem 4.7: Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x)_1 > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (**Figure 4.29**). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

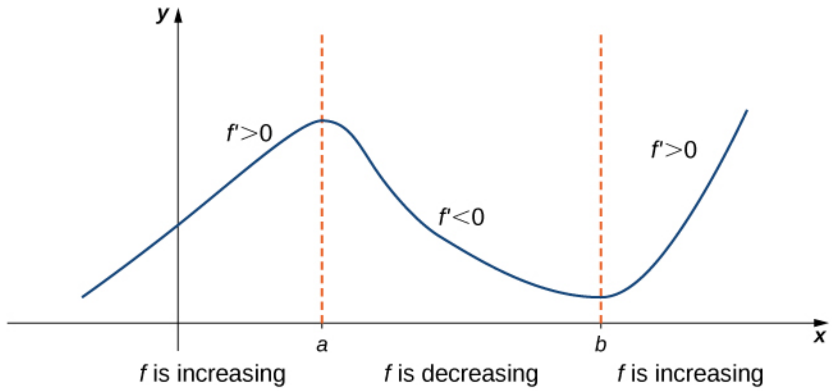


Figure 4.29 If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Theorem 4.8: Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.

ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

Section 4.3: How Derivatives Affect the Shape of a Graph

5.6 Curve Sketching

In this section, we discuss how we can tell what the graph of a function looks like by performing simple tests on its derivatives.

5.6.1. Intervals of Increase/Decrease, and the First Derivative Test

The method of Section 5.2.1 for deciding whether there is a local maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative $f'(x)$ to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? Suppose that f is differentiable at and around $x = a$, and suppose further that a is a critical point of f. Then we have several possibilities:

1. There is a local maximum at $x = a$. This happens if $f'(x) > 0$ as we approach $x = a$ from the left (i.e. when x is in the vicinity of a, and $x < a$) and $f'(x) < 0$ as we move to the right of $x = a$ (i.e. when x is in the vicinity of a, and $x > a$).

2. There is a local minimum at $x = a$. This happens if $f'(x) < 0$ as we approach $x = a$ from the left (i.e. when x is in the vicinity of a, and $x < a$) and $f'(x) > 0$ as we move to the right of $x = a$ (i.e. when x is in the vicinity of a, and $x > a$).

3. There is neither a local maximum or local minimum at $x = a$. If $f'(x)$ does not change from negative to positive, or from positive to negative, as we move from the left of $x = a$ to the right of $x = a$ (that is, $f'(x)$ is positive on both sides of $x = a$, or negative on both sides of $x = a$) then there is neither a maximum nor minimum when $x = a$.

See the first graph in Figure 5.5 and the graph in Figure 5.6 for examples.

Example 5.40: Local Maximum and Minimum

Find all local maximum and minimum points for $f(x) = \sin x + \cos x$ using the first derivative test.

Solution. The derivative is $f'(x) = \cos x - \sin x$ and from Example 5.10 the critical values we need to consider are $\pi/4$ and $5\pi/4$.

We analyze the graphs of $\sin x$ and $\cos x$. Just to the left of $\pi/4$ the cosine is larger than the sine, so $f'(x)$ is positive; just to the right the cosine is smaller than the sine, so $f'(x)$ is negative. This means there is a local maximum at $\pi/4$. Just to the left of $5\pi/4$ the cosine is smaller than the sine, and to the right the cosine is larger than the sine. This means that the derivative $f'(x)$ is negative to the left and positive to the right, so f has a local minimum at $5\pi/4$. ♣

The above observations have obvious intuitive appeal as you examine the graphs in Figures 5.5 and 5.6. We can extend these ideas further and then formulate and prove a theorem: If the graph of f is increasing before (i.e., to the left of) $x = a$ and decreasing after (i.e., to the right of) $x = a$, then there is a local maximum at $x = a$. If the graph of f is decreasing before $x = a$ and increasing after $x = a$, then there is a local minimum at $x = a$. If the graph of f is consistently increasing on either side of $x = a$ or consistently decreasing on either side of $x = a$, then there is neither a local maximum nor a local minimum at $x = a$. We can prove the following theorem using the Mean Value Theorem.

Theorem 5.41: Intervals of Increase and Decrease

If $f'(x) > 0$ for every x in an interval, then f is increasing on that interval.
If $f'(x) < 0$ for every x in an interval, then f is decreasing on that interval.

Proof. We will prove the increasing case. The proof of the decreasing case is similar. Suppose that $f'(x) > 0$ on an interval I. Then f is differentiable, and hence also, continuous on I. If x_1 and x_2 are any two numbers in I and $x_1 < x_2$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem, there is some c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But c must be in I, and thus, since $f'(x) > 0$ for every x in I, $f'(c) > 0$. Also, since $x_1 < x_2$, we have $x_2 - x_1 > 0$. Therefore, both the left hand side and the denominator of the right hand side are positive. It follows that the numerator of the right hand must be positive. That is, $f(x_2) - f(x_1) > 0$, or in other words, $f(x_1) < f(x_2)$. This shows that between x_1 and x_2 in I, the larger one, x_2, necessarily has the larger function value, $f(x_2)$, and the smaller one, x_1, necessarily have the smaller function value, $f(x_1)$. This means that f is increasing on I. ♣

Example 5.42: Local Minimum and Maximum

Consider the function $f(x) = x^4 - 2x^2$. Find where f is increasing and where f is decreasing. Use this information to find the local maximum and minimum points of f.

Solution. We compute $f'(x)$ and analyze its sign.

$$f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x-1)(x+1).$$

The solution of the inequality $f'(x) > 0$ is $(-1,0) \cup (1,\infty)$. So, f is increasing on the interval $(-1,0)$ and on the interval $(1,\infty)$. The solution of the inequality $f'(x) < 0$ is $(-\infty,-1) \cup (0,1)$. So, f is decreasing on the interval $(-\infty,-1)$ and on the interval $(0,1)$. Therefore, at the critical points -1, 0 and 1, respectively, f has a local minimum, a local maximum and a local minimum. ♣

5.6.2. The Second Derivative Test

The basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If f' changes from positive to negative it is decreasing; this means that the derivative of f', f'', might be negative, and if in fact f'' is negative then f' is definitely decreasing. From this we determine that there is a local maximum at the point in question. Note that f' might change from positive to negative while f'' is zero, in which case f'' gives us no information about the critical value. Similarly, if f' changes from negative to positive there is a local minimum at the point, and f' is increasing. If $f'' > 0$ at the point, this tells us that f' is increasing, and so there is a local minimum.

Source: Lyryx Learning; Calculus: Early Transcendentals; 2016, pg 201

Example 5.43: Second Derivative

Consider again $f(x) = \sin x + \cos x$, with $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$. Use the second derivative test to determine which critical points are local maxima or minima.

Solution. Since $f''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$, we know there is a local maximum at $\pi/4$. Since $f''(5\pi/4) = -(-\sqrt{2}/2) - (-\sqrt{2}/2) = \sqrt{2} > 0$, there is a local minimum at $5\pi/4$. ♣

When it works, the second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails, and sometimes the second derivative is quite difficult to evaluate; in such cases we must fall back on one of the previous tests.

Example 5.44: Second Derivative

Let $f(x) = x^4$ and $g(x) = -x^4$. Classify the critical points of $f(x)$ and $g(x)$ as either maximum or minimum.

Solution. The derivatives for $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Zero is the only critical value, but $f''(0) = 0$, so the second derivative test tells us nothing. However, $f(x)$ is positive everywhere except at zero, so clearly $f(x)$ has a local minimum at zero.

On the other hand, for $g(x) = -x^4$, $g'(x) = -4x^3$ and $g''(x) = -12x^2$. So $g(x)$ also has zero as its only critical value, and the second derivative is again zero, but $-x^4$ has a local maximum at zero. ♣

A useful explanation from OpenStax

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.38). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

Source: OpenStax; Calculus Volume 1; 2019, pg 399

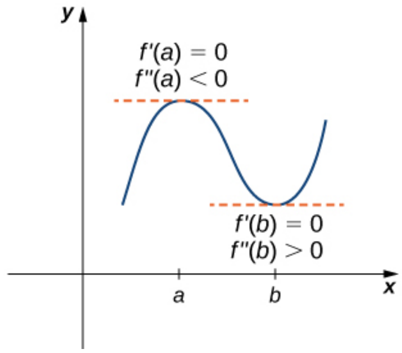


Figure 4.38 Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Theorem 4.11: Second Derivative Test

Suppose $f'(c) = 0$, f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.

ii. If $f''(c) < 0$, then f has a local maximum at c.

iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at x, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.20

Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is

$f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

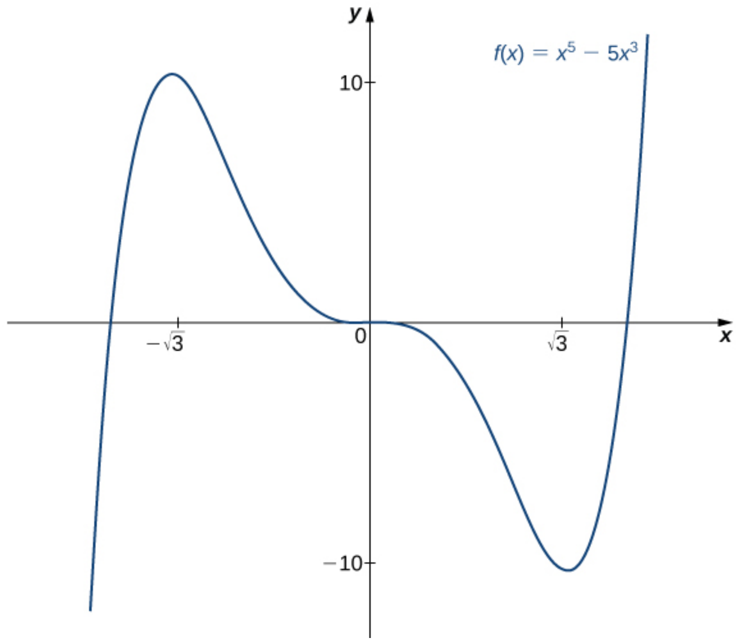


Figure 4.39 The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

4.19 Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

5.6.3. Concavity and Inflection Points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when $f'(x) > 0$, $f(x)$ is increasing. The sign of the second derivative $f''(x)$ tells us whether f' is increasing or decreasing; we have seen that if f' is zero and increasing at a point then there is a local minimum at the point. If f' is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about f from information about f''.

We can get information from the sign of f'' even when f' is not zero. Suppose that $f''(a) > 0$. This means that near $x = a$, f' is increasing. If $f'(a) > 0$, this means that f slopes up and is getting steeper; if $f'(a) < 0$, this means that f slopes down and is getting *less* steep. The two situations are shown in figure 5.14. A curve that is shaped like this is called **concave up**.

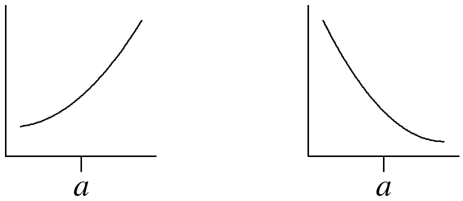


Figure 5.14: $f''(a) > 0$: $f'(a)$ positive and increasing, $f'(a)$ negative and increasing.

Now suppose that $f''(a) < 0$. This means that near $x = a$, f' is decreasing. If $f'(a) > 0$, this means that f slopes up and is getting less steep; if $f'(a) < 0$, this means that f slopes down and is getting steeper. The two situations are shown in figure 5.15. A curve that is shaped like this is called **concave down**.

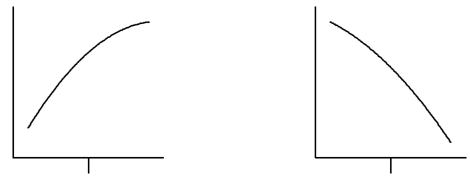


Figure 5.15: $f''(a) < 0$: $f'(a)$ positive and decreasing, $f'(a)$ negative and decreasing.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called **inflection points**. If the concavity changes from up to down at $x = a$, f'' changes from positive to the left of a to negative to the right of a, and usually $f''(a) = 0$. We can identify such points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points. Note that it is possible that $f''(a) = 0$ but the concavity is the same on both sides; $f(x) = x^4$ at $x = 0$ is an example.

Example 5.45: Concavity

Describe the concavity of $f(x) = x^3 - x$.

Solution. The derivatives are $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$. Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from concave down to concave up at zero, and the curve is concave down for all $x < 0$ and concave up for all $x > 0$. ♣

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

Definition

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an **inflection point** of f.

This is another useful example from OpenStax

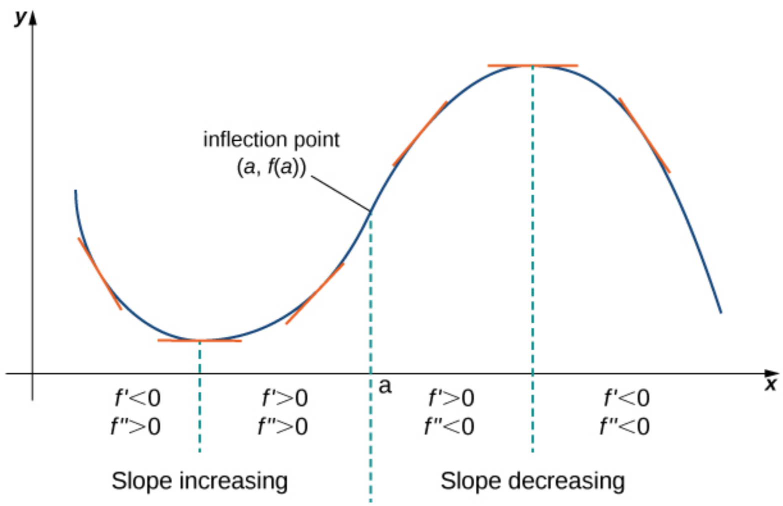


Figure 4.35 Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.19

Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Source: OpenStax; Calculus Volume 1; 2019 , pg 397

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. **Figure 4.36** confirms the analytical results.

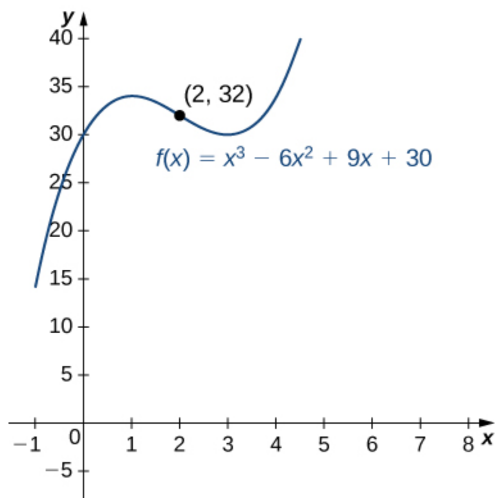


Figure 4.36 The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

4.18 For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

We now summarize, in **Table 4.6**, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in **Figure 4.37**.

This table and graph from OpenStax summarizes the relationship between the derivative tests and concavity

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

Table 4.6 What Derivatives Tell Us about Graphs

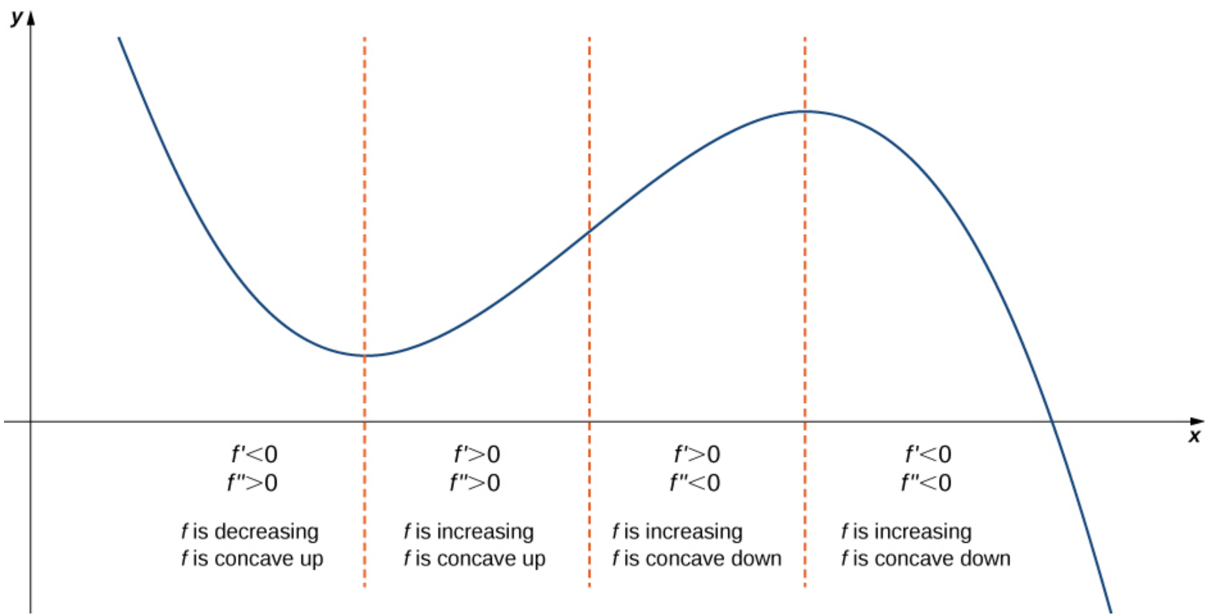


Figure 4.37 Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

5.6.4. Asymptotes and Other Things to Look For

A vertical asymptote is a place where the function becomes infinite, typically because the formula for the function has a denominator that becomes zero. For example, the reciprocal function $f(x) = 1/x$ has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.). Whenever the formula for a function contains a denominator it is worth looking for a vertical asymptote by checking to see if the denominator can ever be zero, and then checking the limit at such points. Note that there is not always a vertical asymptote where the derivative is zero: $f(x) = (\sin x)/x$ has a zero denominator at $x = 0$, but since $\lim_{x\to 0}(\sin x)/x = 1$ there is no asymptote there.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Horizontal asymptotes can be identified by computing the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Since $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$, the line $y = 0$ (that is, the x-axis) is a horizontal asymptote in both directions.

Some functions have asymptotes that are neither horizontal nor vertical, but some other line. Such asymptotes are somewhat more difficult to identify and we will ignore them.

If the domain of the function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. In this case we might also identify this behavior because when $x = \pm r$ the denominator of the function is zero.

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." Its graph is symmetric with respect to the y-axis. Some examples of even functions are: x^n when n is an even number, $\cos x$, and $\sin^2 x$. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. Some examples of odd functions are: x^n when n is an odd number, $\sin x$, and $\tan x$. Of course, most functions are neither even nor odd, and do not have any particular symmetry.

5.6.5. Summary of Curve Sketching

The following is a guideline for sketching a curve $y = f(x)$ by hand. Each item may not be relevant to the function in question, but utilizing this guideline will provide all information needed to make a detailed sketch of the function.

Guideline for Curve Sketching

1. Domain of the function
2. x- and y-Intercepts
3. Symmetry
4. Vertical and Horizontal Asymptotes
5. Intervals of Increase/Decrease, and Local Extrema
6. Concavity and Points of Inflection
7. Sketch the Graph

Example 5.46: Graph Sketching

Sketch the graph of $y = f(x)$ *where* $f(x) = \dfrac{2x^2}{x^2-1}$

Solution.

1. The domain is $\{x : x^2 - 1 \neq 0\} = \{x : x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

2. There is an x-intercept at $x = 0$. The y intercept is $y = 0$.

3. $f(-x) = f(x)$, so f is an even function (symmetric about y-axis)

4. $\displaystyle\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2$, so $y = 2$ is a horizontal asymptote.

   Now the denominator is 0 at $x = \pm 1$, so we compute:

   $$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = +\infty, \ \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty, \ \lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty, \ \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = +\infty.$$

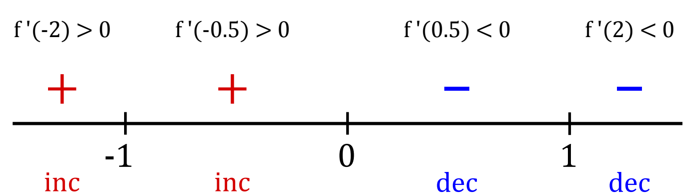
   So the lines $x = 1$ and $x = -1$ are vertical asymptotes.

5. For critical values we take the derivative:

   $$f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

   Note that $f'(x) = 0$ when $x = 0$ (the top is zero). Also, $f'(x) = DNE$ when $x = \pm 1$ (the bottom is zero). As $x = \pm 1$ is *not* in the domain of $f(x)$, the only critical number is $x = 0$ (recall that to be a critical number we need it to be in the domain of the original function).

   Drawing a number line and including *all* of the split points of $f'(x)$ we have:

   Thus f is increasing on $(-\infty, -1) \cup (-1, 0)$ and decreasing on $(0, 1) \cup (1, \infty)$.

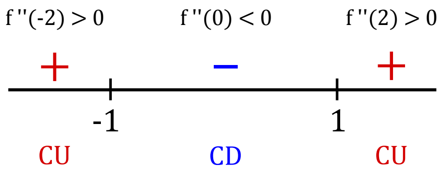
   By the first derivative test, $x = 0$ is a local max.

6. For possible inflection points we take the second derivative:

   $$f''(x) = \frac{12x^2 + 4}{(x^2-1)^3}$$

   The top is never zero. Also, the bottom is only zero when $x = \pm 1$ (neither of which are in the domain of $f(x)$). Thus, there are no possible inflection points to consider.

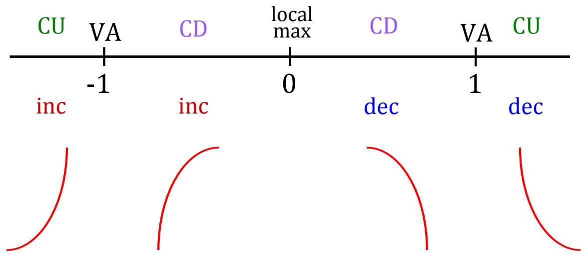
Drawing a number line and including *all* of the split points of $f''(x)$ we have:



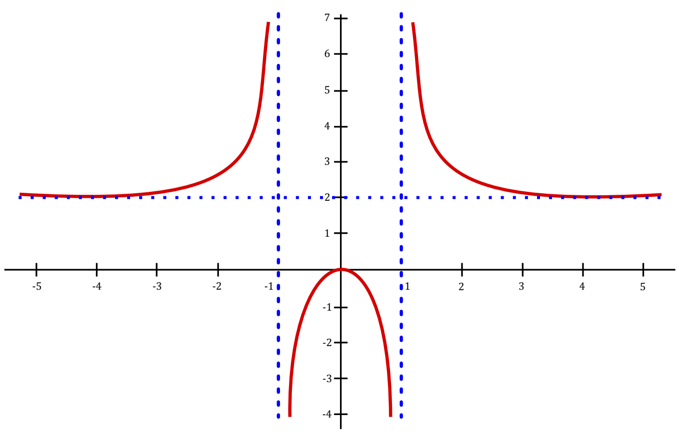
Hence f is concave up on $(-\infty,-1)\cup(1,\infty)$, concave down on $(-1,1)$.

7. We put this information together and sketch the graph.

   We combine some of this information on a single number line to see what *shape* the graph has on certain intervals:



   Note that there is a horizontal asymptote at $y=2$ and that the curve has x-int of $x=0$ and y-int of $y=0$. Therefore, a sketch of $f(x)$ is as follows:



Section 4.4: Indeterminate Forms and l'Hopital's Rule

The following video provides an intuitive explanation of l'Hopital's Rule:
3Blue1Brown - Limits, L'Hopital's rule, and epsilon delta definitions

4.8 | L'Hôpital's Rule

Learning Objectives

- **4.8.1** Recognize when to apply L'Hôpital's rule.
- **4.8.2** Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- **4.8.3** Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2 \neq 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$? We call this one of the **indeterminate forms**, of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \text{ and } \lim_{x \to 0} \frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

For $\lim_{x \to 0} \frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more important, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and such that $g'(a) \neq 0$ For x near $a,$ we can write

$$f(x) \approx f(a) + f'(a)(x - a)$$

and

$$g(x) \approx g(a) + g'(a)(x - a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$

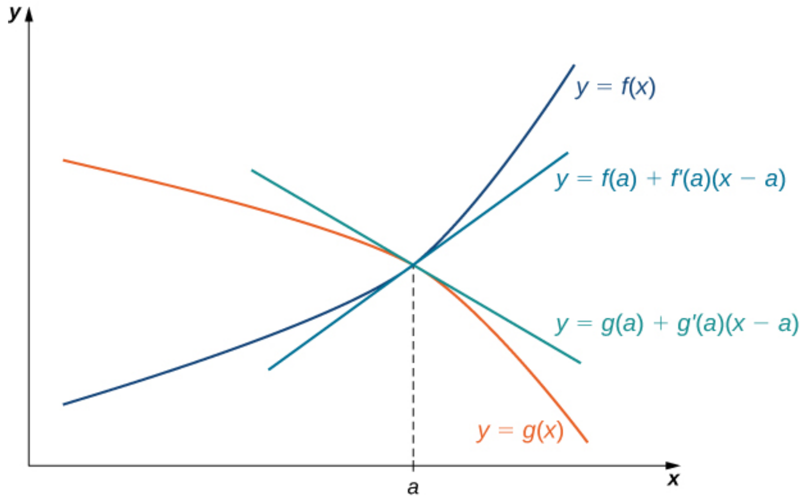


Figure 4.71 If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$ then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at $a,$ then f is continuous at $a,$ and therefore $f(a) = \lim_{x \to a} f(x) = 0.$ Similarly, $g(a) = \lim_{x \to a} g(x) = 0.$ If we also assume that f' and g' are continuous at $x = a,$ then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x).$ Using these ideas, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}.$ Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

Theorem 4.12: L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing $a,$ except possibly at $a.$ If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0,$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty.$ This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty.$

Proof

We provide a proof of this theorem in the special case when $f, g, f',$ and g' are all continuous over an open interval containing $a.$ In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at $a,$ it follows that $f(a) = 0 = g(a).$ Therefore,

$$\begin{aligned}
\lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = 0 = g(a) \\
&= \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} && \text{algebra} \\
&= \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} && \text{limit of a quotient} \\
&= \frac{f'(a)}{g'(a)} && \text{definition of he derivative} \\
&= \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} && \text{continuity of } f' \text{ and } g' \\
&= \lim_{x \to a} \frac{f'(x)}{g'(x)}. && \text{limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.38

Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \to 0} \frac{1 - \cos x}{x}$

b. $\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x}$

c. $\lim_{x \to \infty} \frac{e^{1/x} - 1}{1/x}$

d. $\lim_{x \to 0} \frac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1 - \cos x \to 0$ and the denominator $x \to 0,$ we can apply L'Hôpital's rule to evaluate this limit. We have

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(x)} \\
&= \lim_{x \to 0} \frac{\sin x}{1} \\
&= \frac{\lim_{x \to 0} (\sin x)}{\lim_{x \to 0} (1)} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

b. As $x \to 1,$ the numerator $\sin(\pi x) \to 0$ and the denominator $\ln(x) \to 0.$ Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned} \lim_{x \to 1} \frac{\sin(\pi x)}{\ln x} &= \lim_{x \to 1} \frac{\pi \cos(\pi x)}{1/x} \\ &= \lim_{x \to 1} (\pi x)\cos(\pi x) \\ &= (\pi \cdot 1)(-1) = -\pi. \end{aligned}$$

c. As $x \to \infty,$ the numerator $e^{1/x} - 1 \to 0$ and the denominator $\left(\frac{1}{x}\right) \to 0.$ Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x \to \infty} e^{1/x} = e^0 = 1.$$

d. As $x \to 0,$ both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0,$ we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x \to 0} \frac{\sin x - x}{x^2} = 0.$$

4.37 Evaluate $\lim_{x \to 0} \frac{x}{\tan x}.$

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty.$ Limits of this form are classified as *indeterminate forms of type* $\infty/\infty.$ Again, note that we are not actually dividing ∞ by $\infty.$ Since ∞ is not a real number, that is impossible; rather, $\infty/\infty.$ is used to represent a quotient of limits, each of which is ∞ or $-\infty.$

Theorem 4.13: L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing $a,$ except possibly at $a.$ Suppose $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty.$ This result also holds if the limit is infinite, if $a = \infty$ or

$-\infty$, or the limit is one-sided.

Example 4.39

Applying L'Hôpital's Rule $(\infty/\infty$ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \to \infty} \frac{3x+5}{2x+1}$

b. $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x \to \infty}(3x+5) = \infty$ and $\lim_{x \to \infty}(2x+1) = \infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x \to \infty} \frac{3x+5}{2x+1} = \lim_{x \to \infty} \frac{3}{2} = \frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x \to \infty} \frac{3x+5}{2x+1} = \lim_{x \to \infty} \frac{3+5/x}{2x+1/x} = \frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x \to 0^+} \ln x = -\infty$ and $\lim_{x \to 0^+} \cot x = \infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = \lim_{x \to 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x \to 0^+} \frac{1}{-x\csc^2 x}.$$

Now as $x \to 0^+$, $\csc^2 x \to \infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x \to 0^+} \frac{1}{-x\csc^2 x} = \lim_{x \to 0^+} \frac{\sin^2 x}{-x}.$$

Now $\lim_{x \to 0^+} \sin^2 x = 0$ and $\lim_{x \to 0^+} x = 0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x \to 0^+} \frac{\sin^2 x}{-x} = \lim_{x \to 0^+} \frac{2\sin x \cos x}{-1} = \frac{0}{-1} = 0.$$

We conclude that

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} = 0.$$

4.38 Evaluate $\lim_{x \to \infty} \frac{\ln x}{5x}$.

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example 4.40

When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x \to 1} \frac{x^2+5}{3x+4}$. Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}\left(x^2+5\right) = 2x$$

and

$$\frac{d}{dx}(3x+4) = 3.$$

At which point we would conclude erroneously that

$$\lim_{x \to 1} \frac{x^2+5}{3x+4} = \lim_{x \to 1} \frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x \to 1}\left(x^2+5\right) = 6$ and $\lim_{x \to 1}(3x+4) = 7$, we actually have

$$\lim_{x \to 1} \frac{x^2+5}{3x+4} = \frac{6}{7}.$$

We can conclude that

$$\lim_{x \to 1} \frac{x^2+5}{3x+4} \neq \lim_{x \to 1} \frac{\frac{d}{dx}\left(x^2+5\right)}{\frac{d}{dx}(3x+4)}.$$

4.39 Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$. Evaluate $\lim_{x \to 0^+} \frac{\cos x}{x}$ by other means.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0 \cdot \infty$, $\infty - \infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \to a} (f(x) \cdot g(x))$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to a$. For example, let n be a positive integer and consider

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \to \infty$, $f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \frac{3x^2}{(x^n + 1)}$ varies, depending on n. If $n = 2$, then $\lim_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim_{x \to \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 4.41

Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x \ln x = \frac{\ln x}{1/x},$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\frac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x \to 0^+} x \ln x = 0.$$

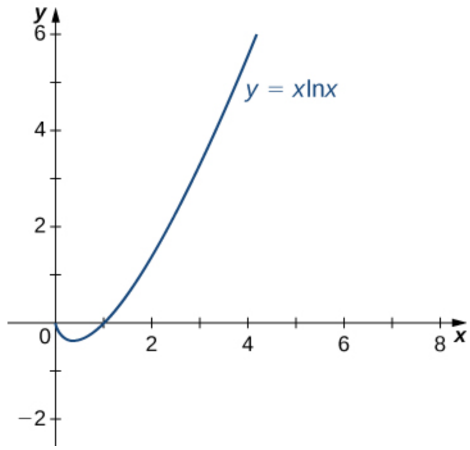


Figure 4.72 Finding the limit at $x = 0$ of the function $f(x) = x\ln x.$

4.40 Evaluate $\lim_{x \to 0} x\cot x.$

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty.$ Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5.$ As $x \to \infty,$ $f(x) \to \infty$ and $g(x) \to \infty.$ We are interested in $\lim_{x \to \infty}(f(x) - g(x)).$ Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty,$ we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3,$ then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x^3 - 3x^2 - 5\right) = \infty.$$

On the other hand, if $n = 2,$ then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x^2 - 3x^2 - 5\right) = -5.$$

However, if $n = 1,$ then

$$\lim_{x \to \infty}(f(x) - g(x)) = \lim_{x \to \infty}\left(3x - 3x^2 - 5\right) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty.$ Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.42

Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{\left(\sec^2 x\right) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $\left(\sec^2 x\right) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{\left(\sec^2 x\right) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right) = \infty.$$

4.41 Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln\left(f(x)^{g(x)}\right) = g(x) \ln(f(x)).$$

Therefore,

$$\lim_{x \to a} [\ln(y)] = \lim_{x \to a} [g(x) \ln(f(x))].$$

Since $\lim_{x \to a} f(x) = \infty$, we know that $\lim_{x \to a} \ln(f(x)) = \infty$. Therefore, $\lim_{x \to a} g(x) \ln(f(x))$ is of the indeterminate form

$0 \cdot \infty,$ and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x \to a} g(x)\ln(f(x)) = L,$ where L may be ∞ or $-\infty$. Then

$$\lim_{x \to a} [\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x \to a} y\right) = L,$$

which gives us

$$\lim_{x \to a} y = \lim_{x \to a} f(x)^{g(x)} = e^L.$$

Example 4.43

Indeterminate Form of Type ∞^0

Evaluate $\lim_{x \to \infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$. Then,

$$\ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x \to \infty} \frac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim_{x \to \infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x \to \infty} y\right) = 0,$$

which leads to

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{\ln x}{x} = e^0 = 1.$$

Hence,

$$\lim_{x \to \infty} x^{1/x} = 1.$$

4.42 Evaluate $\lim_{x \to \infty} x^{1/\ln(x)}$.

Example 4.44

Indeterminate Form of Type 0^0

Evaluate $\lim_{x \to 0^+} x^{\sin x}$.

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x.$$

We now evaluate $\lim_{x \to 0^+} \sin x \ln x$. Since $\lim_{x \to 0^+} \sin x = 0$ and $\lim_{x \to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x \to 0^+} \frac{\cos x}{-1/\left(x(\ln x)^2\right)} = \lim_{x \to 0^+} \left(-x(\ln x)^2 \cos x\right).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x \to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x \to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x \to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim_{x \to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim_{x \to 0^+} y\right) = 0$ and we have

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x \to 0^+} x^{\sin x} = 1.$$

4.43 Evaluate $\lim_{x \to 0^+} x^x$.

Theorem 5.34: L'Hôpital's Rule

For a limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ *of the indeterminate form* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ *if* $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists or equals* ∞ *or* $-\infty$.

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here.

We should also note that there may be instances where we would need to apply L'Hôpital's Rule multiple times, but we must confirm that $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ is still indeterminate before we attempt to apply L'Hôpital's Rule again. Finally, we want to mention that L'Hôpital's rule is also valid for one-sided limits and limits at infinity.

Example 5.35: L'Hôpital's Rule

Compute $\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x}$.

Solution. We use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

$$\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} = \lim_{x \to \pi} \frac{2x}{\cos x},$$

Source: Lyryx Learning; Calculus: Early Transcendentals; 2016, pg 194

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches -1, so

$$\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x} = \frac{2\pi}{-1} = -2\pi.$$

♣

Example 5.36: L'Hôpital's Rule

Compute $\lim_{x\to\infty} \frac{2x^2-3x+7}{x^2+47x+1}$.

Solution. As x goes to infinity, both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{2x^2-3x+7}{x^2+47x+1} = \lim_{x\to\infty} \frac{4x-3}{2x+47}.$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x\to\infty} \frac{4x-3}{2x+47} = \lim_{x\to\infty} \frac{4}{2} = 2.$$

So the original limit is 2 as well. ♣

Example 5.37: L'Hôpital's Rule

Compute $\lim_{x\to 0} \frac{\sec x - 1}{\sin x}$.

Solution. Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sec x - 1}{\sin x} = \lim_{x\to 0} \frac{\sec x \tan x}{\cos x} = \frac{1\cdot 0}{1} = 0.$$

♣

L'Hôpital's rule concerns limits of a quotient that are indeterminate forms. But not all functions are given in the form of a quotient. But all the same, nothing prevents us from re-writing a given function in the form of a quotient. Indeed, some functions whose given form involve either a product $f(x)g(x)$ or a power $f(x)^{g(x)}$ carry indeterminacies such as $0\cdot\pm\infty$ and $1^{\pm\infty}$. Something small times something numerically large (positive or negative) could be anything. It depends on how small and how large each piece turns out to be. A number close to 1 raised to a numerically large (positive or negative) power could be anything. It depends on how close to 1 the base is, whether the base is larger than or smaller than 1, and how large the exponent is (and its sign). We can use suitable algebraic manipulations to relate them to indeterminate quotients. We will illustrate with two examples, first a product and then a power.

Example 5.38: L'Hôpital's Rule

Compute $\lim_{x\to 0^+} x\ln x$.

Solution. This doesn't appear to be suitable for L'Hôpital's Rule, but it also is not "obvious". As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like:

$$(\text{something very small})\cdot(\text{something very large and negative}).$$

This could be anything: it depends on *how small* and *how large* each piece of the function turns out to be. As defined earlier, this is a type of $\pm$"$0\cdot\infty$", which is indeterminate. So we can in fact apply L'Hôpital's Rule after re-writing it in the form $\frac{\infty}{\infty}$:

$$x\ln x = \frac{\ln x}{1/x} = \frac{\ln x}{x^{-1}}.$$

Now as x approaches zero, both the numerator and denominator approach infinity (one $-\infty$ and one $+\infty$, but only the size is important). Using L'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\ln x}{x^{-1}} = \lim_{x\to 0^+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0^+}\frac{1}{x}(-x^2) = \lim_{x\to 0^+} -x = 0.$$

One way to interpret this is that since $\lim_{x\to 0^+} x\ln x = 0$, the x approaches zero much faster than the $\ln x$ approaches $-\infty$. ♣

Finally, we illustrate how a limit of the type "1^∞" can be indeterminate.

Example 5.39: L'Hôpital's Rule

Evaluate $\lim_{x\to 1^+} x^{1/(x-1)}$.

Solution. Plugging in $x=1$ (from the right) gives a limit of the type "1^∞". To deal with this type of limit we will use logarithms. Let

$$L = \lim_{x\to 1^+} x^{1/(x-1)}.$$

Now, take the natural log of both sides:

$$\ln L = \lim_{x\to 1^+} \ln\left(x^{1/(x-1)}\right).$$

Using log properties we have:

$$\ln L = \lim_{x\to 1^+}\frac{\ln x}{x-1}.$$

The right side limit is now of the type $0/0$, therefore, we can apply L'Hôpital's Rule:

$$\ln L = \lim_{x\to 1^+}\frac{\ln x}{x-1} = \lim_{x\to 1^+}\frac{1/x}{1} = 1$$

Thus, $\ln L = 1$ and hence, our original limit (denoted by L) is: $L = e^1 = e$. That is,

$$L = \lim_{x\to 1^+} x^{1/(x-1)} = e.$$

In this case, even though our limit had a type of "1^∞", it actually had a value of e. ♣

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as shown in the following table, the values of x^3 are growing much faster than the values of x^2.

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

Table 4.21 Comparing the Growth Rates of x^2 and x^3

In fact,

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty. \text{ or, equivalently, } \lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in the following table. In fact,

$$\lim_{x \to \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

Table 4.22 Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty; \text{ or, equivalently, } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.45

Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate $\lim_{x \to \infty}\left(\frac{f(x)}{g(x)}\right)$.

a. $f(x) = x^2$ and $g(x) = e^x$

b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x \to \infty} x^2 = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty}\left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x}.$$

Since $\lim_{x \to \infty} 2x = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x \to \infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See **Figure 4.73** and **Table 4.23**).

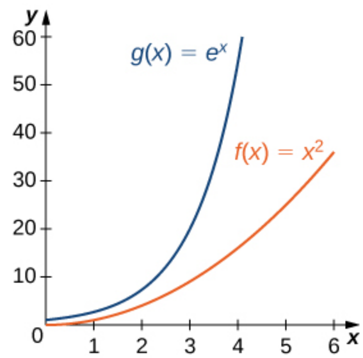


Figure 4.73 An exponential function grows at a faster rate than a power function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

Table 4.23
Growth rates of a power function and an exponential function.

b. Since $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty} \frac{\ln x}{x^2}$. We obtain

$$\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1/x}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see **Figure 4.74** and **Table 4.24**).

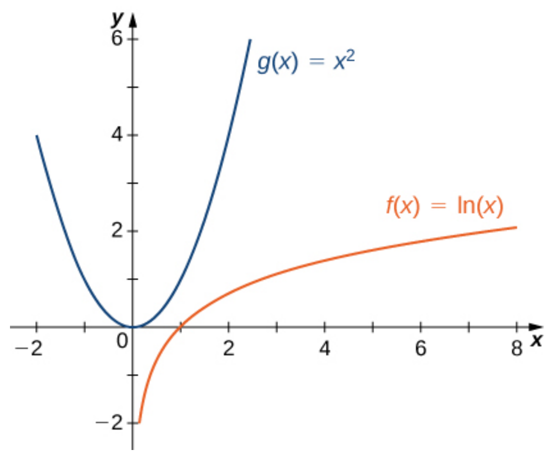


Figure 4.74 A power function grows at a faster rate than a logarithmic function.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

Table 4.24
Growth rates of a power function and a logarithmic function

4.44 Compare the growth rates of x^{100} and 2^x.

Using the same ideas as in **Example 4.45**a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In **Figure 4.75** and **Table 4.25**, we compare e^x with x^3 and x^4 as $x \to \infty$.

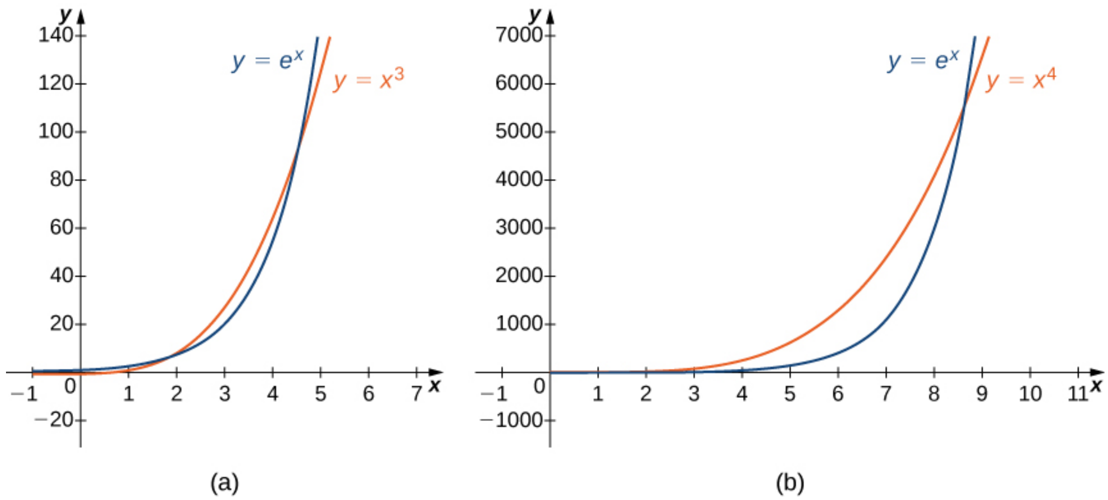


Figure 4.75 The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Table 4.25 An exponential function grows at a faster rate than any power function

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In **Figure 4.76** and **Table 4.26**, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

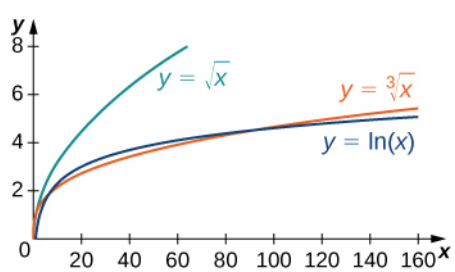


Figure 4.76 The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Table 4.26 A logarithmic function grows at a slower rate than any root function

Section 4.5: Optimization Problems

The following video provides and example of a useful application that was not covered in the open source textbooks:
The Math Sorcerer - Optimization The Closest Point on the Graph

5.7 Optimization Problems

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of $f(x)$ when $a \leq x \leq b$. Sometimes a or b are infinite, but frequently the real world imposes some constraint on the values that x may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions: We are interested only in the function between a and b, and we want to know the largest or smallest value that $f(x)$ takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a *global* (or *absolute*) maximum or minimum.

Guidelines to solving an optimization problem.

1. Understand clearly what is to be maximized or minimized and what the contraints are.
2. Draw a diagram (if appropriate) and label it.
3. Decide what the variables are. For example, A for area, r for radius, C for cost.
4. Write a formula for the function for which you wish to find the maximum or minimum.
5. Express that formula in terms of only one variable, that is, in the form $f(x)$. Usually this is accomplished by using the given constraints.
6. Set $f'(x) = 0$ and solve. Check all critical values and endpoints to determine the extreme value(s) of $f(x)$.

Example 5.47: Largest Rectangle

Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola $y = x^2$ below the line $y = a$ (a is an unspecified constant value), with the top side of the rectangle on the horizontal line $y = a$; see Figure 5.16.)

Solution. We want to find the maximum value of some function $A(x)$ representing area. Perhaps the hardest part of this problem is deciding what x should represent. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. So we can let the x in $A(x)$ be the x of the parabola $f(x) = x^2$. Then the area is

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0,\sqrt{a}]$. (You might object to allowing $x=0$ or $x=\sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting $0 = A'(x) = 6x^2 + 2a$ we get $x = \sqrt{a/3}$ as the only critical value. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. The maximum area thus occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$. ♣

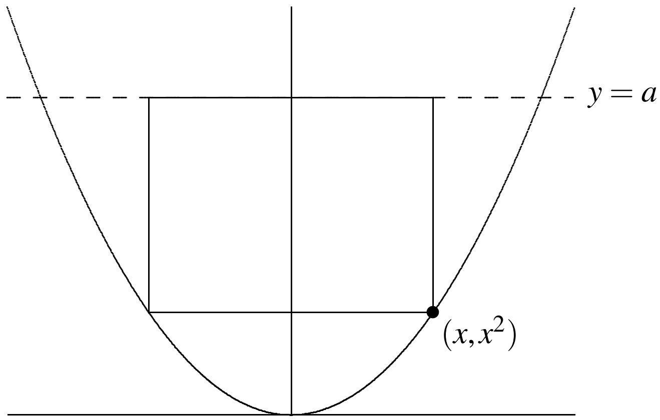


Figure 5.16: Rectangle in a parabola.

Example 5.48: Largest Cone

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Solution. Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\pi r^2 h/3$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible—equal to R—by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the cross-section depicted in Figure 5.17. We have situated the picture in a convenient way relative to the x and y axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $(h-R, r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Either involves taking a square root, but we notice that the volume function contains r^2, not r by itself, so it is easiest to solve for r^2 directly: $r^2 = R^2 - (h-R)^2$.

Then we substitute the result into $\pi r^2 h/3$:

$$\begin{aligned} V(h) &= \pi(R^2-(h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3+\frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V(h)$ when h is between 0 and $2R$. Now we solve $0 = f'(h) = -\pi h^2 + (4/3)\pi hR$, getting $h = 0$ or $h = 4R/3$. We compute $V(0) = V(2R) = 0$ and $V(4R/3) = (32/81)\pi R^3$. The maximum is the latter; since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

♣

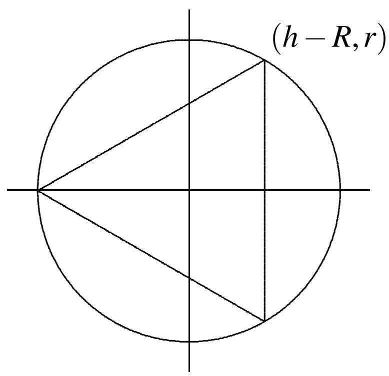


Figure 5.17: Cone in a sphere.

Example 5.49: Containers of Given Volume

You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder.
Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Solution. Let us first choose letters to represent various things: h for the height, r for the base radius, V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder; V and c are constants, h and r are variables. Now we can write the cost of materials:

$$c(2\pi rh) + Nc(2\pi r^2).$$

Again we have two variables; the relationship is provided by the fixed volume of the cylinder: $V = \pi r^2 h$. We use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). The result is

$$f(r) = 2c\pi r\frac{V}{\pi r^2} + 2Nc\pi r^2 = \frac{2cV}{r} + 2Nc\pi r^2.$$

We want to know the minimum value of this function when r is in $(0,\infty)$. We now set $0 = f'(r) = -2cV/r^2 + 4Nc\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Since $f''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi(V/(2N\pi))} = 2N,$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter). ♣

Example 5.50: Rectangles of Given Area

Of all rectangles of area 100, which has the smallest perimeter?

Solution. First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = 2x + 2\frac{100}{x}$$

since the perimeter is twice the length plus twice the width of the rectangle. Not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. Solving $f'(x) = 0$ for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $f'(x)$ is defined everywhere on the interval $(0,\infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is $f''(x) = 400/x^3$, and $f''(10) > 0$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square. ♣

Example 5.51: Maximize your Profit

You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

Solution. The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function $P(x)$ representing the profit when the price per item is x. Profit is revenue minus costs, and revenue is number of items sold times the price per item, so we get $P = nx - 2000 - 0.50n$. The number of items sold is itself a function of x,

$n = 5000 + 1000(1.5 - x)/0.10$, because $(1.5 - x)/0.10$ is the number of multiples of 10 cents that the price is below \$1.50. Now we substitute for n in the profit function:

$$\begin{aligned} P(x) &= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10) \\ &= -10000x^2 + 25000x - 12000 \end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. (Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these.) Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items. ♣

Example 5.52: Minimize Travel Time

Suppose you want to reach a point A that is located across the sand from a nearby road (see Figure 5.18). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Solution. Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean theorem, $\overline{BA} = \sqrt{x^2 + b^2}$, the total time for the trip is

$$f(x) = \frac{a - x}{v} + \frac{\sqrt{x^2 + b^2}}{w}.$$

We want to find the minimum value of f when x is between 0 and a. As usual we set $f'(x) = 0$ and solve for x:

$$\begin{aligned} 0 = f'(x) &= -\frac{1}{v} + \frac{x}{w\sqrt{x^2 + b^2}} \\ w\sqrt{x^2 + b^2} &= vx \\ w^2(x^2 + b^2) &= v^2x^2 \\ w^2b^2 &= (v^2 - w^2)x^2 \\ x &= \frac{wb}{\sqrt{v^2 - w^2}} \end{aligned}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0, a]$, and $wb/\sqrt{v^2 - w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$f''(x) = \frac{b^2}{(x^2 + b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$\begin{aligned} f(0) &= \frac{a}{v}+\frac{b}{w} \\ f(a) &= \frac{\sqrt{a^2+b^2}}{w} \end{aligned}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $f(0) > f(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand. ♣

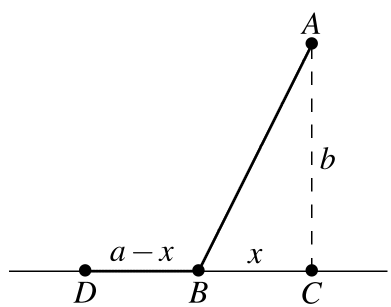


Figure 5.18: Minimizing travel time.

Additional examples from the OpenStax textbook:

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in **Example 4.32**, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 4.32

Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (**Figure 4.62**). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

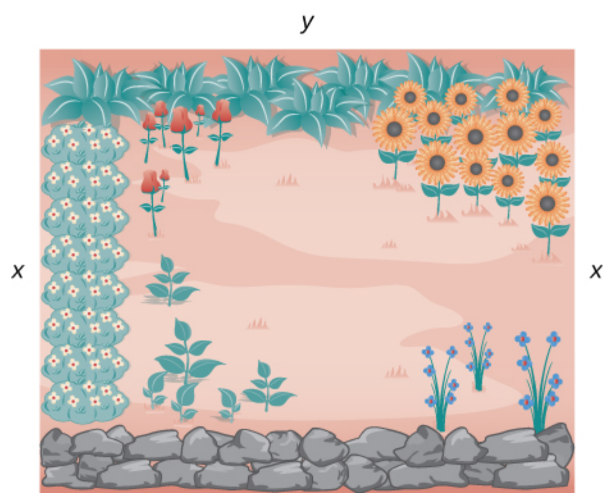


Figure 4.62 We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

Source: OpenStax; Calculus: Volume 1; 2019. pg 439

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From **Figure 4.62**, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (**Figure 4.63**). We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50\text{ ft}$. The area of this garden is 1250 ft^2.

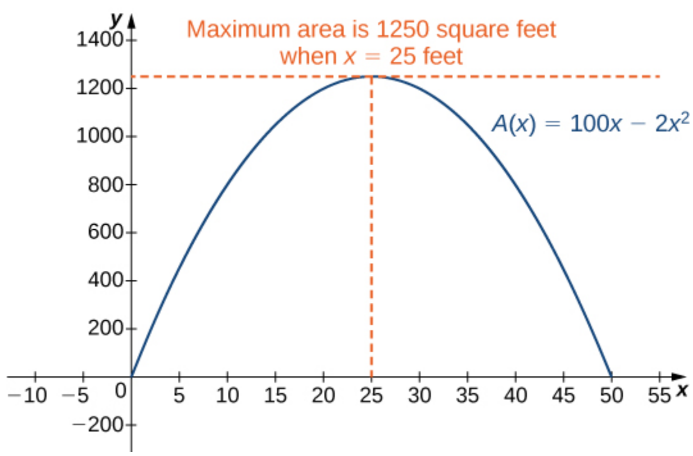


Figure 4.63 To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

4.31 Determine the maximum area if we want to make the same rectangular garden as in **Figure 4.63**, but we have 200 ft of fencing.

Now let's look at a general strategy for solving optimization problems similar to **Example 4.32**.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step $3.$ Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step $4.$ This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.33

Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (**Figure 4.64**). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

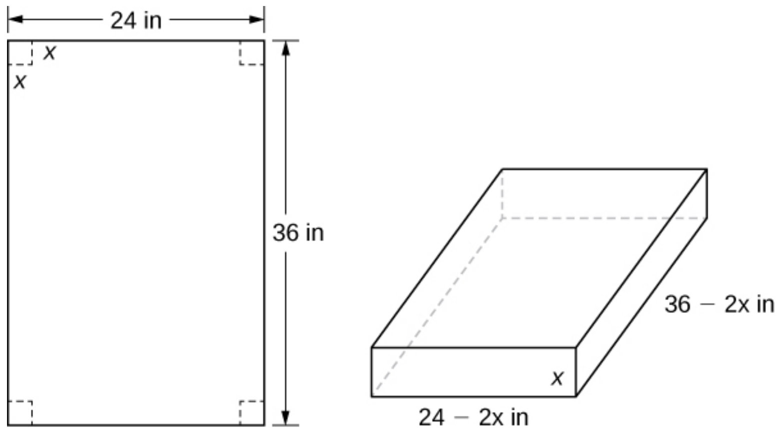


Figure 4.64 A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step $2,$ are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H,$ where $L, W,$ and H are the length, width, and height, respectively.

Step 4: From **Figure 4.64**, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$

Step 5: To determine the domain of consideration, let's examine **Figure 4.64**. Certainly, we need $x > 0.$ Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12).$ Since V is a continuous function over the closed interval $[0, 12],$ we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12],$ V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12,$ the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by $12,$ the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825$ in.^{3} as shown in the following graph.

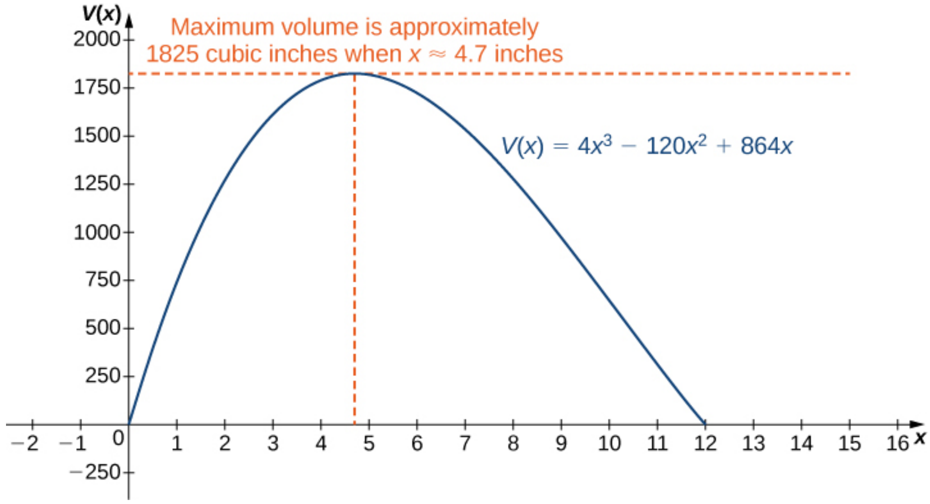


Figure 4.65 Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Watch a **video (http://www.openstaxcollege.org/l/20_boxvolume)** about optimizing the volume of a box.

4.32 Suppose the dimensions of the cardboard in **Example 4.33** are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Example 4.34

Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (**Figure 4.66**). Let T be the time it takes to get from the cabin to the island.

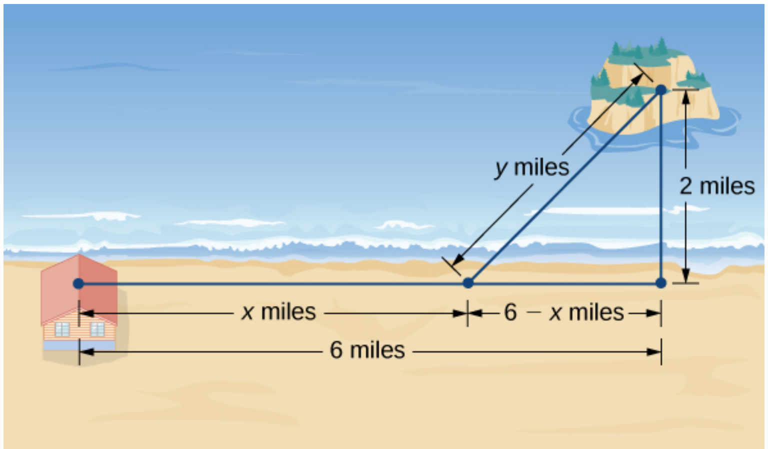


Figure 4.66 How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From **Figure 4.66**, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x\text{ mi}$. Therefore, by the Pythagorean theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain $y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}.$$

Step 5: From **Figure 4.66**, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$T'(x) = \frac{1}{8} - \frac{1}{2}\frac{\left[(6 - x)^2 + 4\right]^{-1/2}}{3} \cdot 2(6 - x) = \frac{1}{8} - \frac{(6 - x)}{3\sqrt{(6 - x)^2 + 4}}.$$

If $T'(x) = 0$, then

$$\frac{1}{8} = \frac{6 - x}{3\sqrt{(6 - x)^2 + 4}}.$$

Therefore,

$$3\sqrt{(6 - x)^2 + 4} = 8(6 - x). \quad (4.6)$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9\left[(6 - x)^2 + 4\right] = 64(6 - x)^2,$$

which implies

$$55(6 - x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x - 6)^2 = \frac{36}{55}.$$

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of **Equation 4.6** to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies **Equation 4.6**. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas $T\left(6 - 6/\sqrt{55}\right) \approx 1.368$ h. Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

4.33 Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

In business, companies are interested in maximizing revenue. In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example 4.35

Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function

$n(p) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$R(p) = n \times p = (1000 - 5p)p = -5p^2 + 1000p.$$

Step 5: Since the owners plan to charge between \$50 per car per day and \$200 per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$ When $p = 100$, $R(100) = \$50{,}000$. When $p = 50$, $R(p) = \$37{,}500$. When $p = 200$, $R(p) = \$0$. Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge \$100 per day per car to maximize revenue as shown in the following figure.

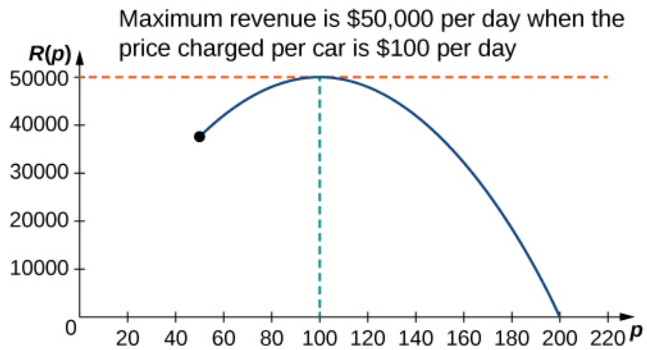


Figure 4.67 To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

4.34 A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Example 4.36

Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

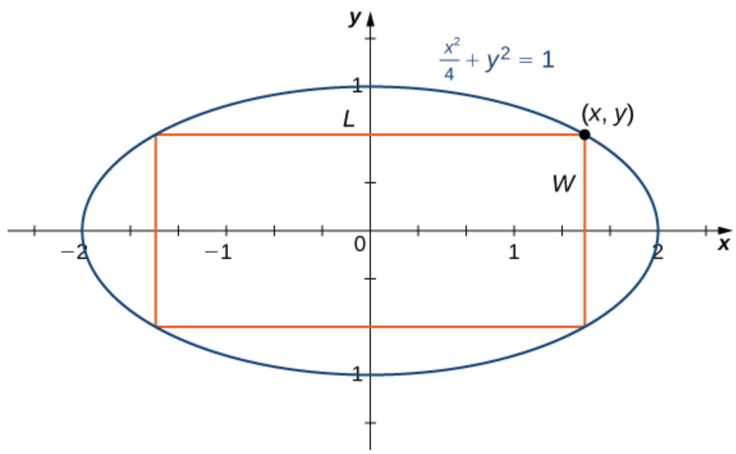


Figure 4.68 We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in **Figure 4.68**. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4 + y^2 = 1}$ and $y > 0$, we have $y = \sqrt{\frac{1 - x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{\frac{1 - x^2}{4}} = 2x\sqrt{4 - x^2}.$$

Step 5: From **Figure 4.68**, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$. Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4-x^2} + 2x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} \\ &= \frac{8-4x^2}{\sqrt{4-x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0.$ We can see that if x is a solution of

$$\frac{8-4x^2}{\sqrt{4-x^2}} = 0, \qquad (4.7)$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2.$ Thus, $x = \pm\sqrt{2}$ are the possible solutions of **Equation 4.7**. Since we are considering x over the interval $[0, 2],$ $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of **Equation 4.7**. Since $x = \sqrt{2}$ is a solution of **Equation 4.7**, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2].$ Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}.$ To determine the dimensions of the rectangle, we need to find the length L and the width $W.$ If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \frac{2}{\sqrt{2}} = \sqrt{2}.$ The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4.$

4.35 Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1.$ What is the domain of consideration?

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0.$ Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty),$ the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example 4.37

Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of 216 in.^3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

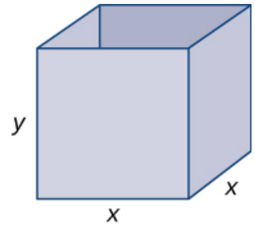


Figure 4.69 We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is $x^2 y$ and the volume is given as 216 in.^3, the constraint equation is

$$x^2 y = 216.$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2 y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2 y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see **Figure 4.70**). When $x = 6\sqrt[3]{2}$, $y = \frac{216}{\left(6\sqrt[3]{2}\right)^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

$$S\left(6\sqrt[3]{2}\right) = \frac{864}{6\sqrt[3]{2}} + \left(6\sqrt[3]{2}\right)^2 = 108\sqrt[3]{4}\text{ in.}^2$$

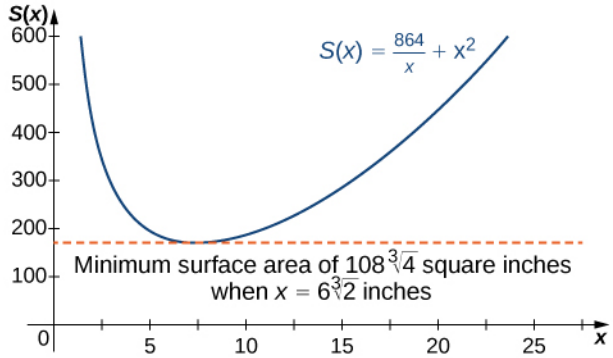


Figure 4.70 We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

4.36 Consider the same open-top box, which is to have volume 216 in.^3. Suppose the cost of the material for the base is $20\,¢/\text{in.}^2$ and the cost of the material for the sides is $30\,¢/\text{in.}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Section 4.6: Newton's Method

The following video provides a useful example that was not covered in the open source textbooks:
Engineer Thileban Explains - Find x3, the third approximation of Newton's method to the root of the equation

4.9 | Newton's Method

Learning Objectives

4.9.1 Describe the steps of Newton's method.
4.9.2 Explain what an iterative process means.
4.9.3 Recognize when Newton's method does not work.
4.9.4 Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0.$ For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the zeroes of functions. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0.$ If f is the first-degree polynomial $f(x) = ax + b,$ then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}.$ If f is the second-degree polynomial $f(x) = ax^2 + bx + c,$ the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0.$ Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0.$ No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0.$ By sketching a graph of $f,$ we can estimate a root of $f(x) = 0.$ Let's call this estimate $x_0.$ We then draw the tangent line to f at $x_0.$ If $f'(x_0) \neq 0,$ this tangent line intersects the x-axis at some point $(x_1, 0).$ Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at $x_1.$ If $f'(x_1) \neq 0,$ this tangent line also intersects the x-axis, producing another approximation, $x_2.$ We continue in this way, deriving a list of approximations: $x_0, x_1, x_2, \ldots.$ Typically, the numbers $x_0, x_1, x_2, \ldots$ quickly approach an actual root $x^*,$ as shown in the following figure.

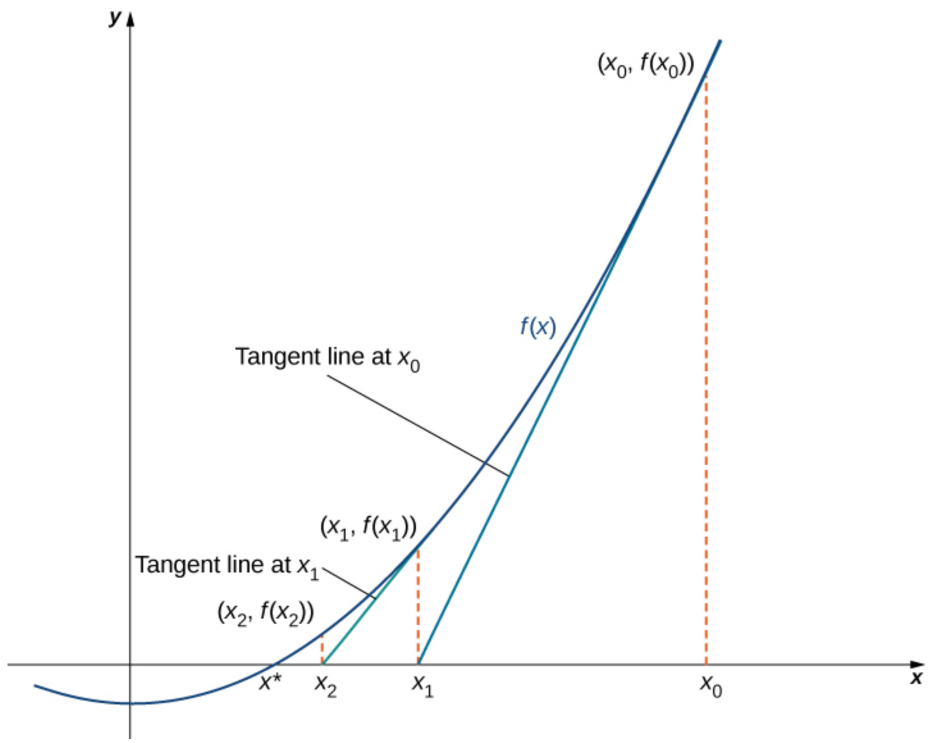


Figure 4.77 The approximations $x_0, x_1, x_2, \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0, x_1, x_2, \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0$, x_n satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \quad \textbf{(4.8)}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 4.46

Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1, x_2, x_3, x_4,$ and x_5.

Solution

From **Figure 4.78**, we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use **Equation 4.8**. Since $f(x) = x^3 - 3x + 1,$ the derivative is $f'(x) = 3x^2 - 3$. Using **Equation 4.8** with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, $x_2,$ we use **Equation 4.8** with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 = \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

$$\begin{aligned} x_1 &\approx 1.666666667 \\ x_2 &\approx 1.548611111 \\ x_3 &\approx 1.532390162 \\ x_4 &\approx 1.532088989 \\ x_5 &\approx 1.532088886 \\ x_6 &\approx 1.532088886. \end{aligned}$$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

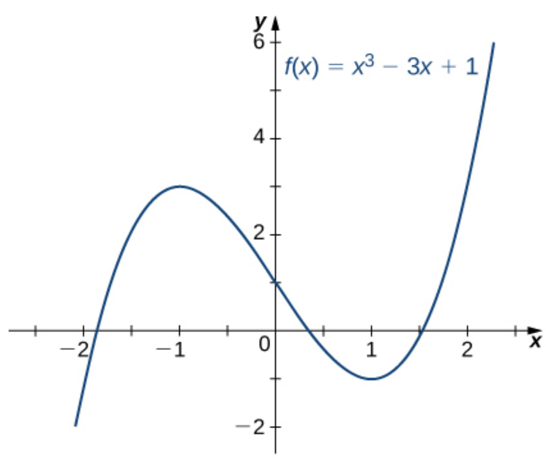


Figure 4.78 The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

4.45 Letting $x_0 = 0,$ let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example 4.47

Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (**Figure 4.79**). Let $f(x) = x^2 - 2,$ let $x_0 = 2,$ and calculate $x_1, x_2, x_3, x_4, x_5.$ (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2},$ the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$.)

Solution

For $f(x) = x^2 - 2,$ $f'(x) = 2x.$ From **Equation 4.8**, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x^2{}_{n-1} - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$x_1 = \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = 1.5$$
$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.416666667.$$

Continuing in this way, we find that

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562 \\ x_5 &\approx 1.414213562. \end{aligned}$$

Since we obtained the same value for x_4 and $x_5,$ it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562.$

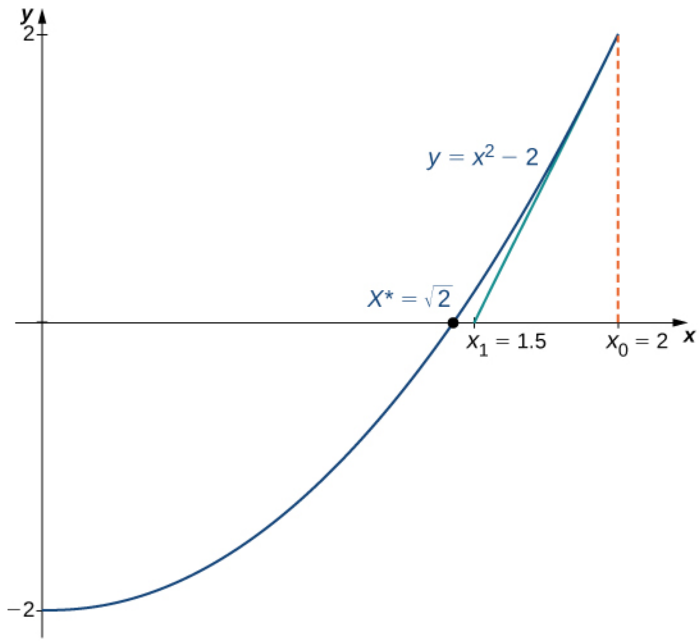


Figure 4.79 We can use Newton's method to find $\sqrt{2}$.

4.46 Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite **Equation 4.8** as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an **iterative process**. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see **Figure 4.80**). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In **Example 4.48**, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

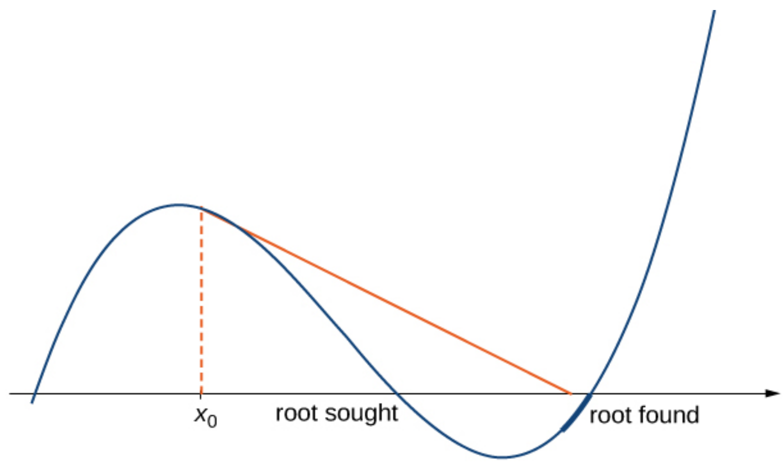


Figure 4.80 If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 4.48

When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2.$ Let $x_0 = 0.$ Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2,$ the derivative is $f'(x) = 3x^2 - 2.$ Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (see **Figure 4.81**). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

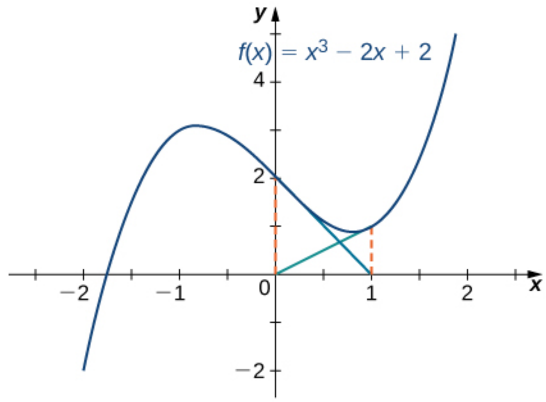


Figure 4.81 The approximations continue to alternate between 0 and 1 and never approach the root of f.

4.47 For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

From **Example 4.48**, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In **Example 4.49**, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 4.49

Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3 \ldots, x_n, \ldots$ as $n \to \infty$. If the list of numbers $x_1, x_2, x_3, \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0$, then

$$x_1 = \frac{1}{2}(0) + 4 = 4$$
$$x_2 = \frac{1}{2}(4) + 4 = 6$$
$$x_3 = \frac{1}{2}(6) + 4 = 7$$
$$x_4 = \frac{1}{2}(7) + 4 = 7.5$$
$$x_5 = \frac{1}{2}(7.5) + 4 = 7.75$$
$$x_6 = \frac{1}{2}(7.75) + 4 = 7.875$$
$$x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$$
$$x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$$
$$x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$$

From this list, we conjecture that the values x_n approach 8.

Figure 4.82 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0, x_1, x_2, \ldots$ approaches $x^* = 8$ as $n \to \infty$.

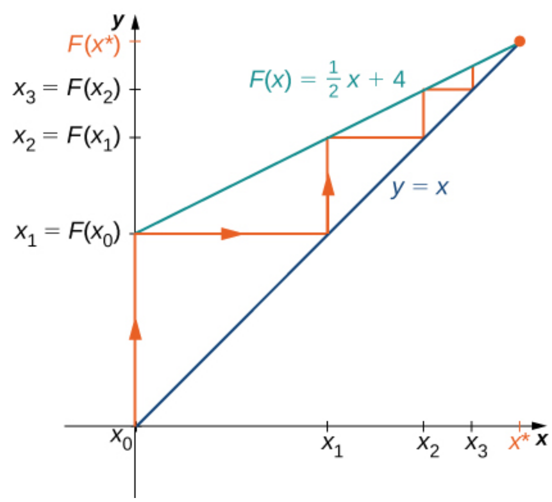


Figure 4.82 This iterative process approaches the value $x^* = 8$.

4.48 Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ as $n \to \infty$.

Section 4.7: Antiderivatives

Rather than stopping at the definition of an antiderivative, the section goes on to introduce integration. This concept will be covered in Calculus 2 (MATH 141), but may also be introduced in MATH 140.

4.10 | Antiderivatives

Learning Objectives

4.10.1 Find the general antiderivative of a given function.
4.10.2 Explain the terms and notation used for an indefinite integral.
4.10.3 State the power rule for integrals.
4.10.4 Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in **Derivatives** of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (**Introduction to Techniques of Integration (http://cnx.org/content/m53654/latest/)**).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F derivative f, we call F an antiderivative of f.

Definition

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x)$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$. Are there any other antiderivatives of f? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

Theorem 4.14: General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

i. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;

ii. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.50

Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \cos x$

d. $f(x) = e^x$

Solution

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln(x)$ and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln(-x)$ and

$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

4.49 Find all antiderivatives of $f(x) = \sin x$.

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is called an *integral sign*, and $\int f(x)dx$ is called the indefinite integral of f.

Definition

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as *integrating* f.

For a function f and an antiderivative F, the functions $F(x) + C$, where C is any real number, is often referred to as *the family of antiderivatives of* f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form $x^2 + C$, we write

$$\int 2x\,dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C,$ where C is any real number, is known as the *family of antiderivatives of* $2x.$ **Figure 4.85** shows a graph of this family of antiderivatives.

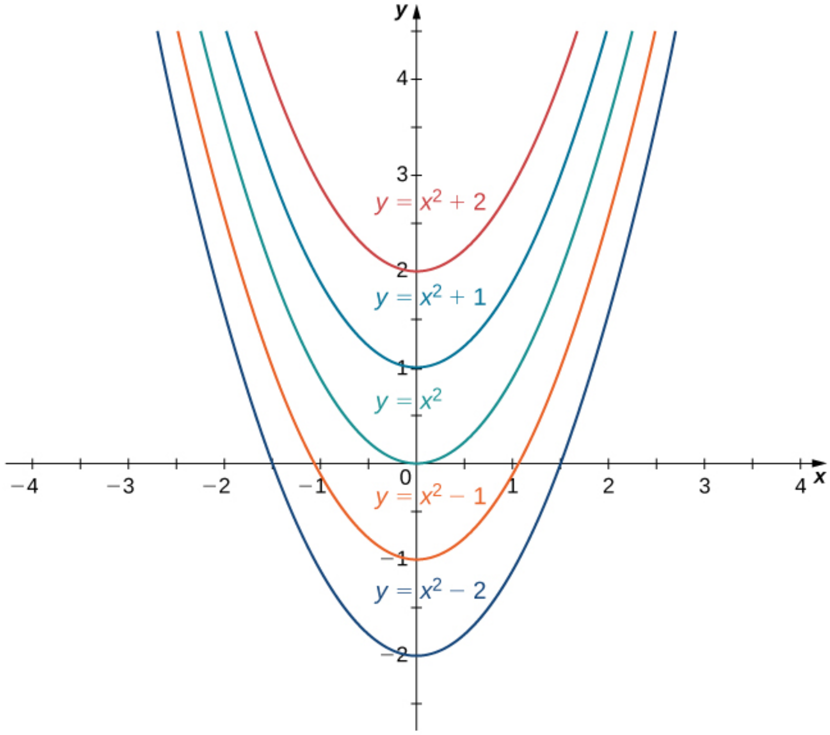


Figure 4.85 The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C,$ where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1,$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n.$$

This fact is known as *the power rule for integrals*.

Theorem 4.15: Power Rule for Integrals

For $n \neq -1,$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in **Appendix B**.

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k dx = \int kx^0 dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dn = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\|x\|) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\|x\| + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1}\|x\|\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}\|x\| + C$

Table 4.27 Integration Formulas

From the definition of indefinite integral of $f,$ we know

$$\int f(x)dx = F(x) + C$$

if and only if F is an antiderivative of f. Therefore, when claiming that

$$\int f(x)dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.51

Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x dx = xe^x - e^x + C$

Solution

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x,$$

the statement

$$\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2 e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2 e^x}{2}\right) = xe^x + \frac{x^2 e^x}{2} \neq xe^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

4.50 Verify that $\int x\cos x\,dx = x\sin x + \cos x + C.$

In **Table 4.27**, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g.$ In **Example 4.51**a. we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\left(\frac{x^2}{2}\right) + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and $g,$ respectively, then

$$\frac{d}{dx}(F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int (f(x) + g(x))dx = F(x) + G(x) + C.$$

Similarly,

$$\int (f(x) - g(x))dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x),$ where k is any real number. Since

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}F(x) = kF'(x)$$

for any real number $k,$ we conclude that

$$\int kf(x)dx = kF(x) + C.$$

These properties are summarized next.

Theorem 4.16: Properties of Indefinite Integrals

Let F and G be antiderivatives of f and $g,$ respectively, and let k be any real number.

Sums and Differences

$$\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see **Example 4.51**b. for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in **Introduction to Integration**. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.52

Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int \left(5x^3 - 7x^2 + 3x + 4\right)dx$

b. $\int \frac{x^2 + 4\sqrt[3]{x}}{x}dx$

c. $\int \frac{4}{1 + x^2}dx$

d. $\int \tan x \cos x\, dx$

Solution

a. Using **Properties of Indefinite Integrals**, we can integrate each of the four terms in the integrand separately. We obtain

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)dx = \int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4dx.$$

From the second part of **Properties of Indefinite Integrals**, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4dx = 5\int x^3\, dx - 7\int x^2\, dx + 3\int x\, dx + 4\int 1dx.$$

Using the power rule for integrals, we conclude that

$$\int \left(5x^3 - 7x^2 + 3x + 4\right)dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int \left(x + \frac{4}{x^{2/3}}\right)dx &= \int x\, dx + 4\int x^{-2/3}dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{\left(\frac{-2}{3}\right) + 1}x^{(-2/3) + 1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using **Properties of Indefinite Integrals**, write the integral as

$$4\int \frac{1}{1 + x^2}dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{\left(1 + x^2\right)}$ to conclude that

$$\int \frac{4}{1 + x^2}dx = 4\tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x}\cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x = \int \sin x = -\cos x + C.$$

4.51 Evaluate $\int (4x^3 - 5x^2 + x - 7)dx.$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A *differential equation* is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x) \quad (4.9)$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of **Equation 4.9** are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2 dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0.$ The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x) \quad (4.10)$$

with the additional condition

$$y(x_0) = y_0 \quad (4.11)$$

is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an *initial condition*. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C,$ to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5.$ From this equation, we see that $C = 3,$ and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

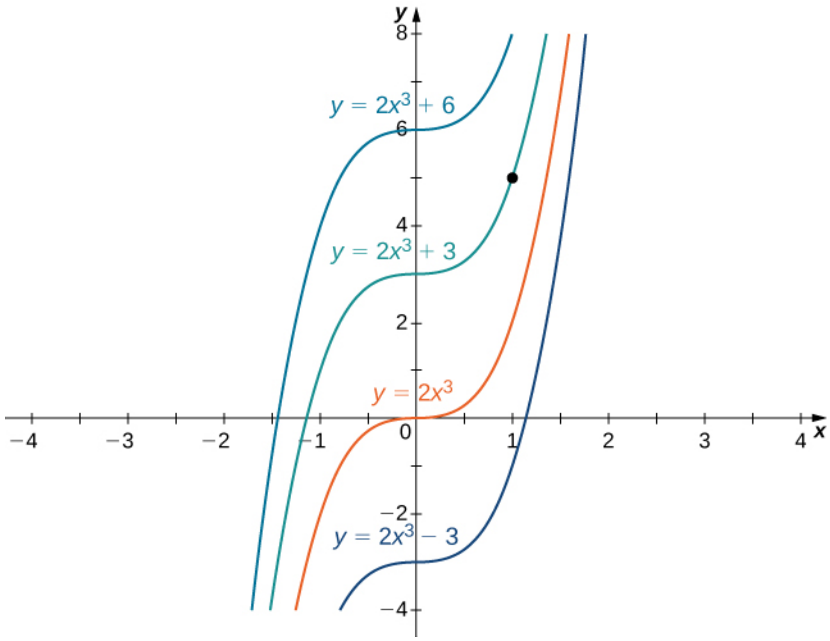


Figure 4.86 Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example 4.53

Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, \; y(0) = 5.$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

$$y = \int \sin(x)dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5$. Therefore,

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6$.

4.52 Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, \; y(1) = 2$.

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car.

We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t),$ and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.54

Decelerating Car

A car is traveling at the rate of 88 ft/sec $(60$ mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.
The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 \text{ ft/s}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, \ v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88,$ $C = 88.$ Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0,$ we obtain $t = \frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s'(t) = -15t + 88, \ s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

4.53 Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?